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# Area-efficient planar straight-line drawings of outerplanar graphs ☆

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**Abstract**

It is important to minimize the area of a drawing of a graph, so that the drawing can fit in a small drawing-space. It is well-known that a planar graph with $n$ vertices admits a planar straight-line grid drawing with $O(n^2)$ area [H. de Fraysseix, J. Pach, R. Pollack, How to draw a planar graph on a grid, Combinatorica 10(1) (1990) 41–51; W. Schnyder, Embedding planar graphs on the grid, in: Proceedings of the First ACM-SIAM Symposium on Discrete Algorithms, 1990, pp. 138–148]. Unfortunately, there is a matching lower-bound of $\Omega(n^2)$ on the area-requirements of the planar straight-line grid drawings of certain planar graphs. Hence, it is important to investigate important categories of planar graphs to determine if they admit planar straight-line grid drawings with $o(n^2)$ area.

In this paper, we investigate an important category of planar graphs, namely, outerplanar graphs. We show that an outerplanar graph $G$ with degree $d$ admits a planar straight-line grid drawing with area $O(dn^{1.48})$ in $O(n)$ time. This implies that if $d = o(n^{0.52})$, then $G$ can be drawn in this manner in $o(n^2)$ area.


*Keywords:* Area-efficient; Outerplanar; Planar; Straight-line

## 1. Introduction

*Graph drawing*: [5,12] focuses on constructing a geometric representation (drawing) of a graph in the plane. It finds applications in several fields, such as software engineering (for visualizing UML models, and function call graphs), databases (for visualizing entity-relationship diagrams), sociology (for visualizing social-networks), and project-management (for visualizing PERT networks). While drawing a graph, it is important to minimize its area, so that the drawing can fit in a limited space, such as a computer screen or printing paper. In this paper, we focus on the problem of constructing drawings with small areas.

We use the graph-drawing-related terms and definitions given in [5]. We review some of them here. Let $G$ be a graph. Two vertices $u$ and $v$ of $G$ are *adjacent* to each other if there is an edge $(u, v)$ in $G$. The vertices that are adjacent to $v$ are called its *neighbors*. The *degree* of $v$ is the number of its neighbors. The *degree* of $G$ is the maximum degree of a vertex. Vertices $u$ and $v$ are called the *end-points* of an edge $(u, v)$.

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A *drawing* $\Gamma$ of $G$ maps each vertex to a distinct point in the plane, and each edge $(u, v)$ to a simple open Jordan curve with endpoints $u$ and $v$. $\Gamma$ is a *straight-line* drawing if each edge is drawn as a single line-segment. $\Gamma$ is a *polyline* drawing if each edge is drawn as a connected sequence of one or more line-segments, where the meeting point of consecutive line-segments is called a *bend*. $\Gamma$ is a *grid* drawing if all the vertices and bends have integer coordinates. $\Gamma$ is a *planar* drawing if edges do not intersect each other. A planar drawing partitions the plane into topologically connected regions called *faces*. The unbounded face is called the *external face*, and the other faces are called *internal* faces. A planar drawing determines a circular ordering on the neighbors of each vertex $v$ according to the clockwise sequence of the incident edges around $v$. Two planar drawings of the same graph are *equivalent* if they determine the same circular orderings of the neighbor sets. A *(planar) embedding* is an equivalence class of planar drawings, and is described by the circular order of the neighbors of each vertex.

Let $\Gamma'$ be a grid drawing. The enclosing rectangle $Encl(\Gamma')$ of $\Gamma'$ is the smallest rectangle with sides parallel to the $X$- and $Y$-axes, respectively, that covers $\Gamma'$ completely. The *width*, *height*, and *area* of $\Gamma'$ are equal to the width, height, and area, respectively, of $Encl(\Gamma')$.

It is well-known that a planar graph with $n$ vertices admits a planar straight-line grid drawing with $\mathrm{O}(n^2)$ area [4,15], and in the worst case it requires $\Omega(n^2)$ area. It is therefore important to investigate important categories of planar graphs to determine if they admit planar straight-line grid drawings with $\mathrm{o}(n^2)$ area.

Outerplanar graphs form an important category of planar graphs. Outerplanar graphs have several applications, such as in computing shortest paths, and network-routing [8–10,13,16]. A graph is *outerplanar* if it is planar and it admits an embedding where all its vertices are on the external face. Currently, the best known upper bound on the area-requirement of a planar straight-line grid drawing of an outerplanar graph with $n$ vertices is $\mathrm{O}(n^2)$, which is the same as for general planar graphs. Hence, a fundamental question arises: can we draw an outerplanar graph in this manner in $\mathrm{o}(n^2)$ area?

In this paper, we give a partial answer to this question by showing that an outerplanar graph $G$ with $n$ vertices and degree $d$ admits a planar straight-line grid drawing with area $\mathrm{O}(dn^{1.48})$ in $\mathrm{O}(n)$ time. This implies that if $d = \mathrm{o}(n^{0.52})$, then $G$ can be drawn in this manner in $\mathrm{o}(n^2)$ area.

From a broader perspective, our contribution is in showing a sufficiently large natural category of planar graphs that can be drawn in $\mathrm{o}(n^2)$ area.

There has been little work done in the past on the area-requirement of grid drawings of outerplanar graphs. Biedl [1] has shown that an outerplanar graph with $n$ vertices admits a planar polyline grid drawing as well as a visibility representation with $\mathrm{O}(n \log n)$ area. Dujmović and Wood [6], and Felsner, Liotta, and Wismath [7] have shown that an outerplanar graph admits an edge-crossing-free straight-line grid drawing with $\mathrm{O}(n)$ volume in the three-dimensional space.

In Section 7, we give our main theorem. It is based on a tree-drawing technique of Chan [2], and uses the fact that the dual of an outerplane graph is a tree. To draw a tree, Chan uses the concept of a special kind of a root-to-leaf path of the tree, called a *spine*. The concept of a spine is useful for drawing outerplane graphs also.

This paper is organized as follows. In Section 2, we review the concept of a spine, as defined by Chan [2]. In Section 3, we define some other concepts that we use in the paper, such as a maximal outerplane graph, the dual tree of a maximal outerplane graph, and a trapezoidal drawing. In Section 4, we present a short overview of our overall strategy for drawing an outerplanar graph. In Section 5, we describe the structure of a maximal outerplane graph with respect to a spine of its dual tree. In Section 6, we explain the construction of a trapezoidal drawing with small area of a maximal outerplane graph. In Section 7, we give our main theorem on drawing outerplanar graphs. Finally, in Section 8, we present our conclusion, and list some open problems.

## 2. Spine of an ordered binary tree

A *leaf* of a tree is a vertex with no child. A *non-leaf* vertex of a tree is one that has at least one child. A *binary* tree is one where each non-leaf vertex has at most two children. An *ordered* tree is a tree with a pre-specified root, and a pre-specified left-to-right ordering of the children of each vertex. In an ordered binary tree, if a node has two children, then they are called its *left* and *right* children, respectively, where the left child precedes the right child in the pre-specified left-to-right ordering. An *empty* tree is one that does not contain any vertex.

Let $T$ be an ordered binary tree. We denote by $|T|$, the number of vertices in $T$. A *subtree* of $T$ *rooted at* a vertex $v$ is the maximal ordered tree that is a subgraph of $T$ and that has $v$ as its root. The *size* of a subtree is equal to the number of vertices in it. A path $P = \pi_0\pi_1 \ldots \pi_a$ of $T$ is a *root-to-leaf* path, if $\pi_0$ is the root of $T$, and $\pi_a$ is a leaf of $T$. Two

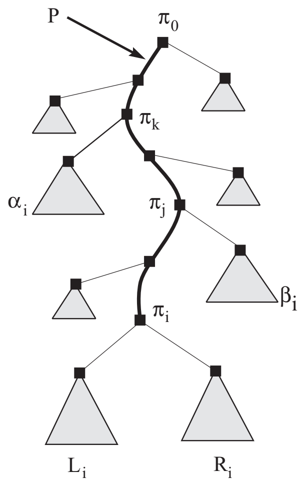


Fig. 1. Subtrees $L_i$, $R_i$, $\alpha_i$, and $\beta_i$ of a spine $P$ of a tree $T$. The edges of $P$ are shown with darker lines. This figure is a minor modification of a similar figure of [2].

vertices of $T$ are *siblings* if they have the same parent. A *subtree of* $P$ is a subtree rooted at the sibling of a vertex of $P$. A subtree of $P$ is a *left* (respectively, *right*) subtree of $P$ if its root is the left (respectively, right) child of its parent. $P$ is the *leftmost* (*rightmost*) path of $T$, if each $\pi_i$, where $1 \leqslant i \leqslant a$, is the left (right) child of its parent.

The following lemma, which paraphrases Lemma A.1 of [2], defines the concept of a *spine*, which is a special kind of a root-to-leaf path.

**Lemma 1** (*Chan [2, Lemma A.1]*). *Let* $p = 0.48$. *In any ordered binary tree* $T$, *there exists a root-to-leaf path* $P$, *called a* spine *of* $T$, *such that for any left subtree* $\alpha$ *and right subtree* $\beta$ *of* $P$, $|\alpha|^p + |\beta|^p \leqslant (1-\delta)|T|^p$, *for some constant* $\delta > 0$. *Also, assuming that we have already pre-computed the size of the subtree rooted at each vertex* $\tau$ *of* $T$ *and stored it in* $\tau$, *we can compute* $P$ *in* $O(|P|)$ *time*.

Chan [2] has given a simple iterative procedure for constructing a spine $P = \pi_0\pi_1 \ldots \pi_a$, as defined in Lemma 1, where each $\pi_i$ is a vertex of $T$. For the sake of completeness, we repeat the procedure here (see Fig. 1). Assume for simplicity that each non-leaf node of $T$ has exactly two children; the procedure can be modified easily to accommodate a non-leaf node that has only one child.

- $\pi_0$ is the same as the root of $T$.
- let $\alpha_i$ ($\beta_i$) be the left (right) subtree with the maximum size among the left (right) subtrees of the path $\pi_0\pi_1 \ldots \pi_i$ (where, $\alpha_0$ and $\beta_0$ are both empty trees). Let $L_i$ and $R_i$ be the subtrees rooted at the left and right children of $\pi_i$, respectively (see Fig. 1).
  - *Case* 1: If $|\alpha_i|^p + |R_i|^p \leqslant (1-\delta)|T|^p$ and $|L_i|^p + |\beta_i|^p > (1-\delta)|T|^p$, then set $\pi_{i+1}$ to be the left child of$\pi_i$.
  - *Case* 2: if $|\alpha_i|^p + |R_i|^p > (1-\delta)|T|^p$ and $|L_i|^p + |\beta_i|^p \leqslant (1-\delta)|T|^p$, set $\pi_{i+1}$ to be the right child of $\pi_i$,
  - *Case* 3: if $|\alpha_i|^p + |R_i|^p \leqslant (1-\delta)|T|^p$ and $|L_i|^p + |\beta_i|^p \leqslant (1-\delta)|T|^p$, we terminate the construction as follows: if $|L_i| \leqslant |R_i|$, set the spine to be the concatenation of path $\pi_0\pi_1 \ldots \pi_i$ and the leftmost path from $\pi_i$ to a leaf $\pi_a$, otherwise (i.e. $|L_i| > |R_i|$), set the spine to be the concatenation of the path $\pi_0\pi_1 \ldots \pi_i$ and the rightmost path from $\pi_i$ to a leaf $\pi_a$.
  - *Case* 4: $|\alpha_i|^p + |R_i|^p > (1-\delta)|T|^p$ and $|L_i|^p + |\beta_i|^p > (1-\delta)|T|^p$. Chan [2] has proved that this case is not possible for $p = 0.48$ (with a sufficiently small value of $\delta$).

**Remark 1.** The reason why $p$ has value 0.48 in Lemma 1 is because for $p = 0.48$, Case 4 cannot happen, and so, there will always exist a spine $P$ as defined. This is proved by Chan [2]. The crux of his proof is as follows: Let $\pi_k$ and $\pi_j$ be the parents of the roots of $\alpha_i$ and $\beta_i$, respectively (see Fig. 1). Assume without loss of generality that $j > k$. Since $\beta_i$ is the right subtree of $\pi_j$, by the construction of $P$, it follows that Case 1 applies to $\pi_j$. So, $|L_j|^p + |\beta_j|^p > (1-\delta)|T|^p$. If Case 4 were to happen, then $2.5(1-\delta)|T|^p < |\alpha_i|^p + |R_i|^p + |\beta_i|^p + |L_i|^p + 0.5|L_j|^p + 0.5|\beta_j|^p$. Using additional inequalities, $|L_i| + |R_i| \leqslant |L_j|$, and $|\alpha_i| + |\beta_i| + |L_j| + |\beta_j| \leqslant |T|$, and with some mathematical manipulation, Chan shows that if Case 4 were to happen, then $2.5(1-\delta)|T|^p < (2+(2^{1-p}+0.5)^{1/(1-p)}+0.5^{1/(1-p)})^{1-p}|T|^p$. For $p=0.48$, this would imply that $2.5(1-\delta)|T|^p < 2.499|T|^p$. For a sufficiently small value of $\delta$, $2.5(1-\delta)|T|^p \geqslant 2.499|T|^p$. So, for $p = 0.48$, with a sufficiently small value of $\delta$, $2.5(1-\delta)|T|^p < 2.499|T|^p$ cannot happen, and so, Case 4 cannot happen.

**Remark 2.** If $|L_i|$ and $|R_i|$ are already pre-computed and stored in the left and right child, respectively, of $\pi_i$, then they can be accessed in O(1) time, assuming that we already have access to $\pi_i$. Moreover, $\alpha_i$ and $\beta_i$ can be computed easily from $\alpha_{i-1}$ and $\beta_{i-1}$, respectively, in O(1) time: if $\pi_i$ is the left child of $\pi_{i-1}$, then $\alpha_i = \alpha_{i-1}$, and $\beta_i$ is the subtree with the greater size among $\beta_{i-1}$ and the subtree rooted at the sibling of $\pi_i$; if $\pi_i$ is the right child of $\pi_{i-1}$, then $\beta_i = \beta_{i-1}$, and $\alpha_i$ is the subtree with the greater size among $\alpha_{i-1}$ and the subtree rooted at the sibling of $\pi_i$. So, overall, iteration $i$ takes only O(1) time to execute in Cases 1 and 2. Iteration $i$ takes $\mathrm{O}(1) + \mathrm{O}(|P| - i) = \mathrm{O}(|P| - i)$ time to execute in Case 3, but the procedure terminates at the end of the iteration. So, overall, the procedure spends O(1) time per vertex of $P$. Hence, the procedure will take at most $|P| \cdot \mathrm{O}(1) = \mathrm{O}(|P|)$ time to construct $P$.

The following lemma is a minor modification of a similar result of Chan [2]:

**Lemma 2.** *Let $d'$, $p$, and $\delta$ be three numbers with $0 < p < 1$, $0 < \delta < 1$, and $d' > 0$. Let $\Psi(m)$, where $m \geqslant 0$ is an integer, denote the set of all the pairs of integers such that for each pair $(m_1, m_2) \in \Psi(m)$, $0 \leqslant m_1, m_2 \leqslant m-1$ and $m_1^p + m_2^p \leqslant (1-\delta)m^p$, i.e., $\Psi(m) = \{(m_1, m_2) | 0 \leqslant m_1, m_2 \leqslant m-1 \text{ and } m_1^p + m_2^p \leqslant (1-\delta)m^p\}$. Let $\phi$ be a function over non-negative integers such that*

$$\begin{aligned} \phi(m) &= \mathrm{O}(1) \quad \text{for } m = 0, \\ &\leqslant \max_{(m_1,m_2)\in\Psi(m)} \{\phi(m_1) + \phi(m_2) + d'\} \quad \text{for } m \geqslant 1, \end{aligned}$$

*then, $\phi(m) = \mathrm{O}(d'm^p)$.*

**Proof.** We will use induction over $m$ to prove that $\phi(m) = \mathrm{O}(d'm^p)$. We will show that there exist constants $c_1$ and $c_2$ such that $\phi(m) \leqslant c_1 d' m^p + c_2$.

If $m=0$, then $\phi(m) = \mathrm{O}(1) \leqslant c'$, for some constant $c'$. For a sufficiently large value of $c_2$, $\phi(0) \leqslant c' \leqslant c_2 = c_1 d' \cdot 0^p + c_2$.

Now assume that $m \geqslant 1$. Since $\delta < 1$, $\Psi(m)$ is non-empty because it contains at least one pair, namely, (0, 0). Let $(m_1, m_2)$ be a pair of $\Psi(m)$. Hence, $m_1, m_2 \leqslant m-1$, and $m_1^p + m_2^p \leqslant (1-\delta)m^p$. From the inductive hypothesis, $\phi(m_1) \leqslant c_1 d' m_1^p + c_2$, and $\phi(m_2) \leqslant c_1 d' m_2^p + c_2$. Hence, $\phi(m_1) + \phi(m_2) + d \leqslant c_1 d' m_1^p + c_2 + c_1 d' m_2^p + c_2 + d' = d'\{c_1(m_1^p + m_2^p) + 1 + 2c_2/d'\}$. Since $m_1^p + m_2^p \leqslant (1-\delta)m^p$, $\phi(m_1) + \phi(m_2) + d' \leqslant d'\{c_1(1-\delta)m^p + 1 + 2c_2/d'\} = d'\{c_1 m^p - c_1\delta m^p + 1 + 2c_2/d'\} = d'\{c_1 m^p + c_2/d' - c_1\delta m^p + 1 + c_2/d'\}$.

For a sufficiently large value of $c_1$, $c_1\delta m^p \geqslant 1 + c_2/d'$, and so, $\phi(m_1) + \phi(m_2) + d' \leqslant d'(c_1 m^p + c_2/d') = c_1 d' m^p + c_2$. Hence, $\phi(m) \leqslant c_1 d' m^p + c_2$ (for a sufficiently large value of $c_1$). $\square$

## 3. Preliminaries

A graph is *connected* if there is a path between $u$ and $v$ for each pair $(u, v)$ of vertices. A *cutvertex* of a graph is a vertex whose removal disconnects the graph. A *biconnected* graph is one that does not contain any cutvertex. A *biconnected component* of a graph $G$ is a maximal biconnected subgraph of $G$. Throughout this paper, we will assume that a graph is connected. We will denote by $|G|$, the number of vertices in a graph $G$. We denote by $|S|$, the number of elements of a set $S$.

A *maximal* outerplanar graph is one to which no edge can be added without destroying its outerplanarity. An *outerplane* graph is an embedding of an outerplanar graph where all the vertices are on the external face. It is easy to see that a maximal outerplane graph is biconnected, and each internal face of the graph is bounded by a 3-cycle, i.e., a cycle consisting of exactly three edges. The reverse is also true, i.e., if an outerplane graph is biconnected, and each internal face is bounded by a 3-cycle, then it is a maximal outerplane graph.

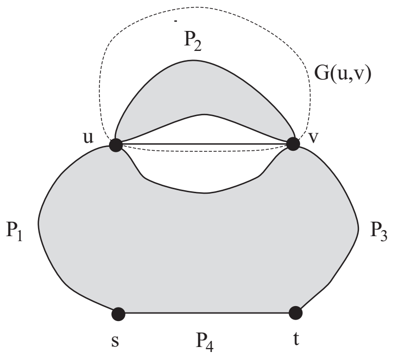


Fig. 2. Subgraph $G(u, v)$, where $(u, v) \neq (s, t)$, of a graph $G(s, t)$. In this figure, $G(u, v)$ is the graph enclosed by the dotted closed-curve.

Let $G$ be a maximal outerplane graph. Let $s$ and $t$ be two distinct distinguished adjacent vertices of $G$; edge $(s, t)$ is called the *reference* edge of $G$.

Let $G(u, v)$, where $(u, v)$ is an edge of $G$, denote the maximal outerplane graph that is defined as follows:

- if $(u, v) = (s, t)$ then $G(u, v) = G$, and
- if $(u, v) \neq (s, t)$, then $G(u, v)$ is defined as follows: Assume that the clockwise order of $s, t, u$, and $v$ on the boundary of $G$ is $s, u, v, t$. The boundary of $G$ can be partitioned into four paths, $P_1$, $P_2$, $P_3$, and $P_4$, where $P_1$ connects $s$ and $u$, $P_2$ connects $u$ and $v$, $P_3$ connects $v$ and $t$, and $P_4$ consists of only one edge, namely, $(s, t)$ (see Fig. 2). Note that $P_2$ consists of only one edge, namely, $(u, v)$, if $(u, v)$ is on the external face. Also, if $s = u$ (respectively, $t = v$), then $P_1$ (respectively, $P_3$) consists of only one vertex, namely, $s$ (respectively, $t$).
  $G(u, v)$ is the subgraph of $G$ whose boundary consists of the path $P_2$ and edge $(u, v)$ (see Fig. 2). For example, Fig. 4(a) shows $G(o_1, o_2)$ for the edge $(o_1, o_2)$ of the graph $G$ shown in Fig. 3(a).

Since $G(u, v) = G$ if $(u, v) = (s, t)$, we can refer to $G$ as $G(s, t)$.

Let $x(v)$ and $y(v)$ denote the $x$- and $y$-coordinates, respectively, of a vertex $v$ in a drawing $\Gamma'$ of $G(s, t)$. The pair $(a, b)$ denotes the line-segment that connects two points $a$ and $b$ in the plane. We denote the width and height of $\Gamma'$ by $W(\Gamma')$ and $H(\Gamma')$, respectively.

A *trapezoidal* drawing of $G(s, t)$ is a planar straight-line grid drawing where:

- $|y(t) - y(s)| \leqslant 1$, $x(t) - x(s) = |G(s, t)| - 1$, and
- for each vertex $o$, where $o \neq s, t$, we have that $y(o) > \min\{y(s), y(t)\}$, and $x(s) < x(o) < x(t)$.

**Lemma 3.** *Let $\Gamma$ be a trapezoidal drawing of $G(s, t)$, then no vertex other than $s$ and $t$ is on the left or right boundary of Encl$(\Gamma)$.*

**Proof.** Let $o$ be a vertex of $G(s, t)$, where $o \neq s, t$. We have that $x(s) < x(o) < x(t)$. Hence, $o$ cannot be on the left or right boundary of *Encl*$(\Gamma)$. □

The *dual tree* $T(s, t)$ of $G(s, t)$ is an ordered binary tree that is defined as follows:

- If $G(s, t)$ consists of only one edge, namely, $(s, t)$, then $T(s, t)$ is the empty tree, i.e., the tree containing no vertex.
- If $G(s, t)$ contains at least two edges, then it contains at least one internal face.
  - Corresponding to each internal face $f$ of $G$, $T(s, t)$ contains a vertex $\tau(f)$.
  - Corresponding to each edge $e = (u, v)$ of $G(s, t)$, such that $e$ is not on the external face, $T(s, t)$ contains an edge $e' = (\tau(f_1), \tau(f_2))$, where $f_1$ and $f_2$ are the two internal faces of $G(s, t)$ whose common boundary consists of $e$. Edges $e$ and $e'$ are called the *duals* of each other.
  - $T(s, t)$ is rooted at the internal face of $G(s, t)$ that has $(s, t)$ on its boundary.
  - For each edge $(\tau_1, \tau_2)$ of $T(s, t)$, $\tau_1$ is the parent of $\tau_2$ if and only if $\tau_1$ is on the unique path of $T(s, t)$ that connects $\tau_2$ and the root of $T(s, t)$.

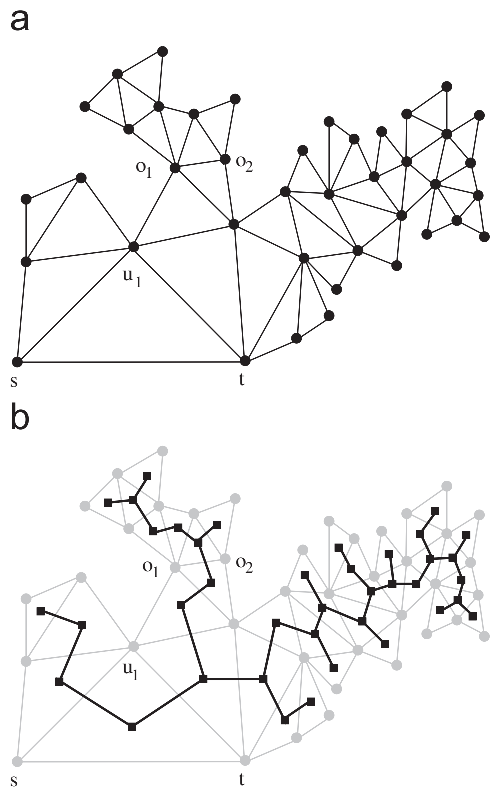


Fig. 3. (a) A graph $G(s,t)$. (b) The dual tree $T(s,t)$ of $G(s,t)$. In part (b) the edges of $T(s,t)$ are shown with darker lines.

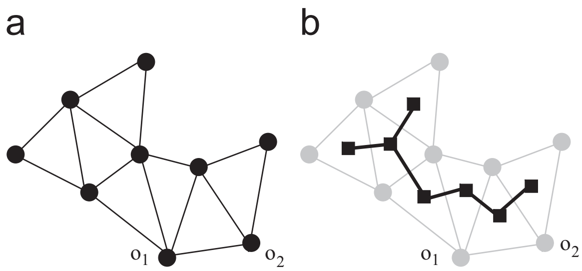


Fig. 4. (a) Subgraph $G(o_1,o_2)$ of the graph $G(s,t)$ of Fig. 3(a). (b) The dual tree $T(o_1,o_2)$ of $G(o_1,o_2)$; in part (b) the edges of $T(o_1,o_2)$ are shown with darker lines.

- For each non-leaf vertex $\tau(f)$ of $T(s,t)$, its left and right children are defined as follows: face $f$ has exactly three edges $e_1$, $e_2$, and $e_3$, on its boundary. Let $e_1, e_2, e_3$ be the counterclockwise sequence of these edges on the boundary of $f$. We have two cases depending upon whether $\tau(f)$ is the root of $T(s,t)$. If $\tau(f)$ is not the root of $T(s,t)$, then let $\tau_1$, $\tau_2$, and $\tau_3$ be the neighbors of $\tau(f)$ in $T(s,t)$, such that the edge $(\tau(f),\tau_i)$, where $1 \leqslant i \leqslant 3$, is the dual of the edge $e_i$. Assume without loss of generality that $\tau_1$ is the parent of $\tau(f)$. Then, $\tau_2$ is the left child of $\tau(f)$, and $\tau_3$ is the right child of $\tau_f$. If $\tau(f)$ is the root of $T(s,t)$, then assume without loss of generality that $e_1=(s,t)$. Let $\tau_2$ and $\tau_3$ be the neighbors of $\tau(f)$ in $T(s,t)$, such that the edges $(\tau(f),\tau_2)$ and $(\tau(f),\tau_3)$ are the duals of the edge $e_2$ and $e_3$, respectively. Then, $\tau_2$ is the left child of $\tau(f)$, and $\tau_3$ is the right child of $\tau(f)$.

For example, Fig. 3(b) shows the dual tree $T(s,t)$ of the maximal outerplane graph $G(s,t)$ shown in Fig. 3(a). Note that $T(s,t)$ is rooted at the vertex corresponding to the internal face that has $s$ and $t$ on its boundary (for the graph $G(s,t)$ of Fig. 3(a), it is the face that has $s$, $t$, and $u_1$ on its boundary).

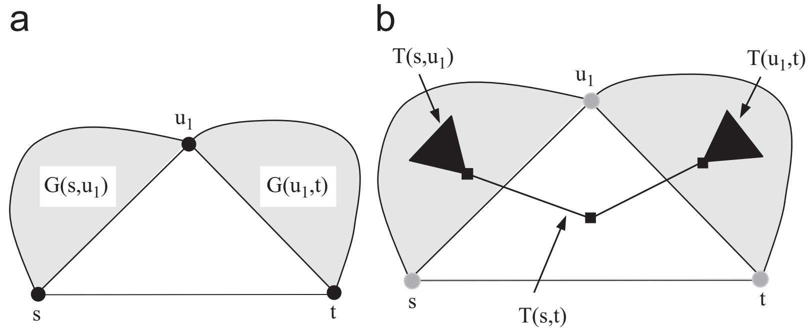


Fig. 5. (a) Structure of $G(s,t)$ if it contains at least two internal faces. (b) Dual tree $T(s,t)$ of $G(s,t)$; the edges of $T(s,t)$ are shown with darker lines.

The boundary of each internal face of $G(s,t)$ consists of exactly three edges. Hence, each non-leaf vertex of $T(s,t)$ has exactly two children.

It is easy to see that the dual tree $T(u,v)$ of $G(u,v)$, where $(u,v)$ is an edge of $G(s,t)$, is the subtree of $T(s,t)$ that is rooted at the vertex corresponding to the internal face of $G(u,v)$ that has $(u,v)$ on its boundary. For example, Fig. 4(b) shows $T(o_1,o_2)$ for the graph $G(o_1,o_2)$ of Fig. 4(a).

The following lemma is immediate:

**Lemma 4.** *Tree $T(s,t)$ consists of $\mathrm{O}(|G(s,t)|)$ vertices, and can be constructed in $\mathrm{O}(|G(s,t)|)$ time.*

## 4. Overview of our drawing strategy

Let $G$ be an outerplanar graph with degree $d$. Our overall strategy for drawing $G$ is very simple (see the proof of Theorem 1 for a more formal description). First, we convert $G$ into a maximal outerplane graph $G^*$ by inserting sufficient number of edges into it, such that $|G^*| = \mathrm{O}(|G|)$ and $d^* = \mathrm{O}(d)$, where $d^*$ is the degree of $G^*$. Next, we construct a trapezoidal drawing $\Gamma^*$ of $G^*$. Finally, from $\Gamma^*$ we obtain a drawing of $G$ by removing the edges that were inserted into $G$ to convert it into $G^*$.

To construct $\Gamma^*$, we designate an arbitrarily selected edge $(s,t)$ on the external face of $G^*$ to be its reference edge, and construct $\Gamma^*$ recursively using a divide-and-conquer approach (note that in the proof of Theorem 1, $\Gamma^*$ is called formally as $\Gamma^*(s,t,0)$):

- A spine $P^*$ of the dual tree of $G^*(s,t)$ is computed. Let $H^*$ be the subgraph of $G^*(s,t)$ whose faces correspond to the nodes of $P^*$. Graph $G^*(s,t)$ is splitted into several smaller maximal outerplane graphs, called its *child graphs*, by removing the vertices and edges of $H^*$, and some other vertices and edges from it. The child graphs are determined by interpreting the structure of $G^*(s,t)$ with respect to $P^*$ (for details, see Section 5).
- A trapezoidal drawing of each child graph is constructed recursively. Then, these drawings are combined together appropriately, and the vertices and edges that were removed from $G^*(s,t)$ to obtain the child graphs are also placed back appropriately to obtain $\Gamma^*$ (for details, see Section 6).

**Remark 3.** $\Gamma^*$ is a trapezoidal drawing. Hence, its width is equal to $|G^*(s,t)| - 1 = \mathrm{O}(|G^*(s,t)|)$. Moreover, its height is $\mathrm{O}(d^* \cdot |G^*(s,t)|^p)$, where $p = 0.48$ (see the proof of Lemma 7). Hence, $\Gamma^*$ has area $\mathrm{O}(d^* \cdot |G^*(s,t)|^{1+p}) = \mathrm{O}(d^* \cdot |G^*(s,t)|^{1.48}) = \mathrm{O}(d \cdot |G|^{1.48})$, since $d^* = \mathrm{O}(d)$ and $|G^*| = \mathrm{O}(|G|)$. Hence, the drawing of $G$ also has area $\mathrm{O}(d \cdot |G|^{1.48})$.

## 5. Structure of a maximal outerplane graph $G(s,t)$

Let $T(s,t)$ be the dual tree of $G(s,t)$. Let $P$ be a spine of $T(s,t)$. We can describe the structure of $G(s,t)$ with respect to $P$ as follows: If $G(s,t)$ consists of exactly one edge, namely, $(s,t)$, or if it consists of exactly one internal face (which has $(s,t)$ on its boundary), then we are done. Otherwise, assume that $s$ precedes $t$ in the counterclockwise

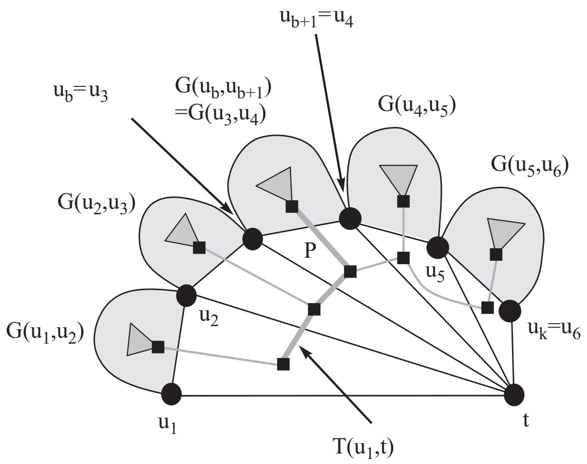


Fig. 6. Structure of $G(u_1, t)$. Here $b = 3$ and $k = 6$. Also shown is its dual tree $T(u_1, t)$ and path $P$. The edges of $T(u_1, t)$ are shown with gray lines. The edges of $P$ are shown with thick gray lines.

ordering of the vertices on the external face of $G(s, t)$. Let $u_1$ be the common neighbor of $s$ and $t$ (see Fig. 5). $G(s, t)$ contains two maximal outerplanar graphs $G(s, u_1)$ and $G(u_1, t)$.

Exactly one edge among $(s, u_1)$ and $(u_1, t)$ is the dual of an edge of $P$; assume that $(u_1, t)$ is that edge (we can describe the structure of $G(s, t)$ in a symmetrical fashion if $(s, u_1)$ is that edge).

Since $(u_1, t)$ is the dual of an edge $e'_1$ of $P$, $G(u_1, t)$ will contain at least one internal face (which corresponds to an end-point of $e'_1$).

### 5.1. Structure of $G(u_1, t)$

We can describe the structure of $G(u_1, t)$ as follows: Let $u_1, u_2, \ldots, u_k$ be the clockwise sequence of the neighbors of $t$ in $G(u_1, t)$. $G(u_1, t)$ contains $k - 1$ maximal outerplanar graphs, $G(u_1, u_2), G(u_2, u_3), \ldots, G(u_{k-1}, u_k)$, that are arranged in a clockwise order as shown in Fig. 6. Note that $G(u_1, t)$ also contains $G(u_k, t)$, which consists of only one edge, namely, $(u_k, t)$.

Among the edges $(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k)$, where $u_{k+1} = t$, at most one edge is the dual of an edge of $P$; let $b$ be the integer such that:

- if none of these edges is the dual of an edge of $P$, then $b = k$,
- otherwise, $(u_b, u_{b+1})$ is the edge that is the dual of an edge of $P$.

### 5.2. Structure of $G(u_b, u_{b+1})$, where $1 \leqslant b \leqslant k - 1$

Since $1 \leqslant b \leqslant k - 1$, $(u_b, u_{b+1})$ is the dual of an edge $e'_2$ of $P$. $G(u_b, u_{b+1})$ contains at least one internal face (which corresponds to an end-point of $e'_2$). Hence, $u_b$ (respectively, $u_{b+1}$) has at least one neighbor in $G(u_b, u_{b+1})$ that is different from $u_{b+1}$ (respectively, $u_b$). Let $v_1, v_2, \ldots, v_m$ be the ordered sequence of the neighbors of $u_b$ and $u_{b+1}$ in $G(u_b, u_{b+1})$, such that (see Fig. 7)

- for each $i$, where $1 \leqslant i \leqslant m$, $v_i \neq u_b, u_{b+1}$,
- the neighbors of $u_b$ are placed in clockwise order,
- the neighbors of $u_{b+1}$ are placed in clockwise order, and
- the neighbors of $u_b$ precede the neighbors of $u_{b+1}$.

Graph $G(u_b, u_{b+1})$ contains $m - 1$ maximal outerplane graphs, $G(v_1, v_2), G(v_2, v_3), \ldots, G(v_{m-1}, v_m)$, that are arranged in a clockwise order as shown in Fig. 7(a). Note that $G(u_b, u_{b+1})$ also contains two subgraphs $G(u_b, v_1)$

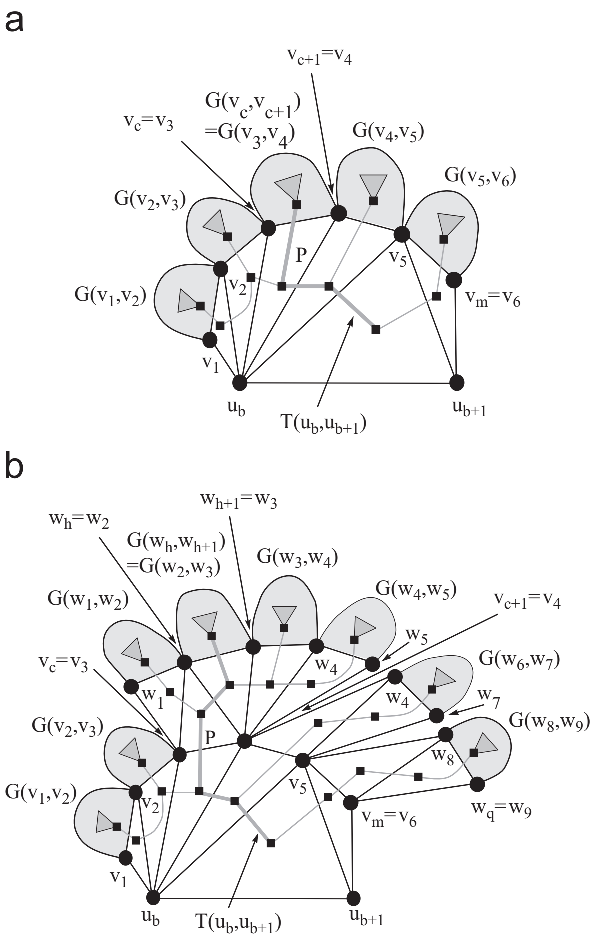


Fig. 7. (a) Structure of a $G(u_b, u_{b+1})$, where $1 \leqslant b \leqslant k-1$. Here $c = 3$ and $m = 6$. (b) Structure of $G(u_b, u_{b+1})$ in terms of $G(v_1, v_2), G(v_2, v_3), \ldots, G(v_{c-1}, v_c)$, and $G(w_1, w_2), G(w_2, w_3), \ldots, G(w_{q-1}, w_q)$. Here, $h = 2$ and $q = 9$. In both Parts (a) and (b), we also show $T(u_b, u_{b+1})$ and $P$. The edges of $T(u_b, u_{b+1})$ are shown with gray lines, and the edges of $P$ are shown with thick gray lines.

and $G(v_m, u_{b+1})$, both of which contain only one edge each, namely, $(u_b, v_1)$ and $(v_m, u_{b+1})$, respectively. Among the edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)$, at most one edge is the dual of an edge of $P$; let $c$ be the integer that is defined as follows:

- If none of these edges is the dual of an edge of $P$, then there can be two cases.
  - If $m = 1$, then $G(u_b, u_{b+1})$ contains only one internal face $f$. The boundary of $f$ consists of $u_b$, $u_{b+1}$, and $v_1$. The leaf of $P$ corresponds to $f$. In this case, $c = 0$.
  - If $m \geqslant 2$, then we have two subcases.
    If the leaf of $P$ corresponds to the face whose boundary contains $u_b$, $v_1$, and $v_2$, then $c = 0$.
    Otherwise, the leaf of $P$ corresponds to the face whose boundary contains $u_{b+1}$, $v_{m-1}$, and $v_m$. Then, $c = m$.
- Otherwise, $c$ is the integer such that $(v_c, v_{c+1})$ is the dual of an edge of $P$ (see Fig. 7(a)).

If $c = 0$, then for convenience, we will define $v_0 = u_b$. If $c = m$, i.e., if $v_c = v_m$, then for convenience, we will define $v_{m+1} = u_{b+1}$.

Let $w_1, w_2, \ldots, w_q$ be the (possibly empty) ordered sequence of the neighbors of $v_c, v_{c+1}, \ldots, v_m$ in the graph $G' = G(v_c, v_{c+1}) \cup G(v_{c+1}, v_{c+2}) \cup \cdots \cup G(v_{m-1}, v_m)$, such that (see Fig. 7(b))

- for each $i$, where $1 \leqslant i \leqslant q$, $w_i \neq v_c, v_{c+1}, \ldots, v_m$,

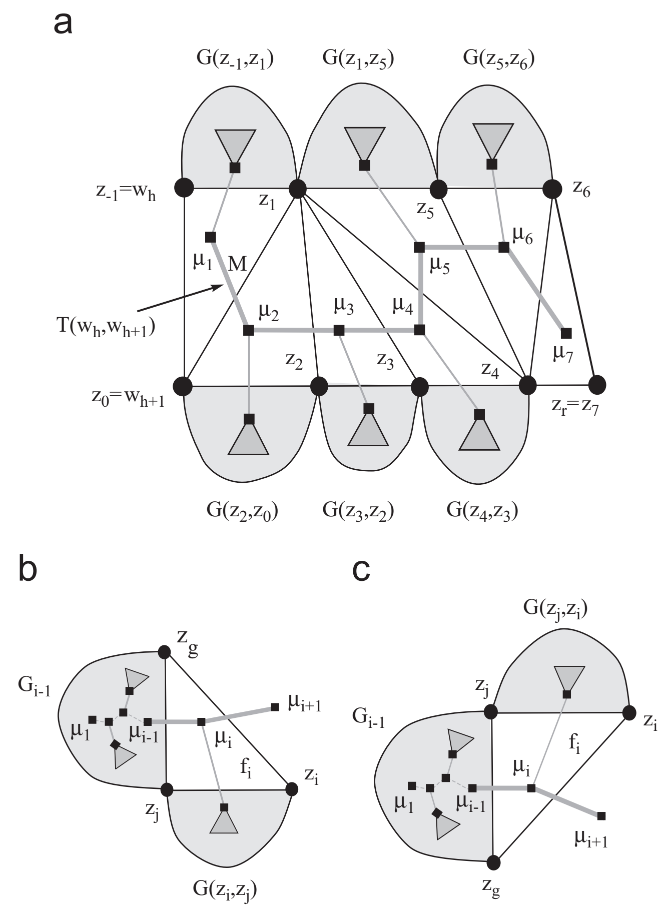


Fig. 8. (a) Structure of a $G(w_h, w_{h+1})$, where $(w_h, w_{h+1})$ is the dual of an edge of $P$. Here, $r = 7$. Also shown are $T(w_h, w_{h+1})$ and $M$. The edges of $T(w_h, w_{h+1})$ are shown with gray lines. The edges of $M$ are shown with thick gray lines. (b) Graph $G_i$ if $\mu_{i+1}$ is the right child of $\mu_i$. (c) Graph $G_i$ if $\mu_{i+1}$ is the left child of $\mu_i$.

- for each $i$, where $c \leqslant i \leqslant m$, the neighbors (if any) of $v_i$ are placed in clockwise order, and
- for each $i$, where $c \leqslant i \leqslant m - 1$, the neighbors (if any) of $v_i$ precede the neighbors (if any) of $v_{i+1}$.

For each pair of vertices $w_i$ and $w_{i+1}$, where $1 \leqslant i \leqslant q-1$, there are two possibilities: either there is an edge $(w_i, w_{i+1})$ connecting them, or there is none. If there is an edge $(w_i, w_{i+1})$, then $G(u_b, u_{b+1})$ contains a maximal outerplanar graph $G(w_i, w_{i+1})$. Hence, $G(u_b, u_{b+1})$ contains at most $q - 1$ maximal outerplane graphs of the form $G(w_i, w_{i+1})$ that are arranged in a clockwise order as shown in Fig. 7(b). Note that $G(u_b, u_{b+1})$ also contains two subgraphs $G(v_c, w_1)$ and $G(w_q, v_{c+1})$, which consist of only one edge each, namely, $(v_c, w_1)$ and $(w_q, v_{c+1})$, respectively.

If $q = 0$, then we are done, otherwise among the edges $(w_1, w_2), (w_2, w_3), \ldots, (w_{q-1}, w_q)$, at most one edge is the dual of an edge of $P$; let $h$ be the integer that is defined as follows:

- *Case* I: If none of these edges is the dual of an edge of $P$, then there can be three cases.
  - If $c = 0$ or $q = 1$, then $h = 0$. (Note that if $q = 1$, then $G'$ consists of only one internal face; the leaf of $P$ will correspond to this face.)
  - If the leaf of $P$ corresponds to the face whose boundary contains $v_c$, $w_1$, and $w_2$, then $h = 0$.
  - Otherwise, $h$ is the integer such that the leaf of $P$ corresponds to the face whose boundary contains $v_{c+1}$, $w_{h-1}$, and $w_h$.
- *Case* II: Otherwise, $h$ is the integer such that $(w_h, w_{h+1})$ is the dual of an edge of $P$ (see Fig. 7(b)).

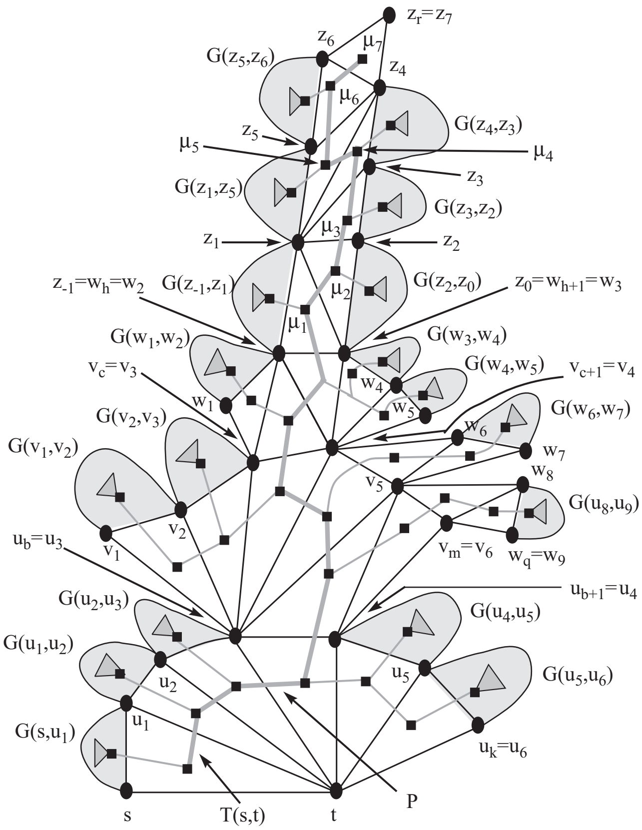


Fig. 9. Structure of a $G(s,t)$ with respect to a spine $P$ in the general case where $b \neq k$, $m \geqslant 2$, $1 \leqslant c \leqslant m-1$, and Case II applies to $G(u_b, u_{b+1})$. Also shown are $T(s,t)$ and $P$. The edges of $T(s,t)$ are shown with gray lines. The edges of $P$ are shown with thick gray lines.

If Case I applies to $G(u_b, u_{b+1})$, then we are done. Otherwise, we can describe the structure of $G(w_h, w_{h+1})$ as in Section 5.3.

### 5.3. *Structure of $G(w_h, w_{h+1})$, where $(w_h, w_{h+1})$ is the dual of an edge of $P$*

Since $(w_h, w_{h+1})$ is the dual of an edge $e'_3$ of $P$, $G(w_h, w_{h+1})$ contains at least one internal face (which corresponds to an end-point of $e'_3$). Let $M = \mu_1 \mu_2 \ldots \mu_r$ be the maximal subpath of $P$ such that each edge $(\mu_i, \mu_{i+1})$, where $1 \leqslant i \leqslant r-1$, is the dual of an edge of $G(w_h, w_{h+1})$ (see Fig. 8(a)). Note that $\mu_1$ will correspond to the internal face of $G(w_h, w_{h+1})$ that has $(w_h, w_{h+1})$ on its boundary.

We can describe the structure of $G(w_h, w_{h+1})$ as a sequence of graphs $G_0, G_1, \ldots, G_r$, where $G(w_h, w_{h+1}) = G_r$, and for each $i$, where $0 \leqslant i \leqslant r-1$, $G_i \subset G_{i+1}$. We will also correspondingly define vertices $z_{-1}, z_0, z_1, \ldots, z_r$ of $G(w_h, w_{h+1})$.

- Graph $G_0$ consists of only one edge $(w_h, w_{h+1})$. Vertex $z_{-1} = w_h$ and $z_0 = w_{h+1}$.
- For each $i$, where $1 \leqslant i \leqslant r$, $G_i$ is defined as follows (see Fig. 8(b,c)).
  - If $1 \leqslant i \leqslant r-1$, then let $f_i$ be the internal face of $G(w_h, w_{h+1})$ that corresponds to $\mu_i$. Vertex $z_i$ is the vertex on the boundary of $f_i$ that is not in $G_{i-1}$. Let $z_j$ and $z_g$ be the other two vertices on the boundary of $f_i$. Both $z_j$ and $z_g$ are in $G_{i-1}$ and $j, g < i$. Assume without loss of generality that $(\mu_i, \mu_{i+1})$ is the dual of $(z_g, z_i)$.
    If $\mu_{i+1}$ is the right child of $\mu_i$ (see Fig. 8(b)), then $G_i = G_{i-1} \cup G(z_i, z_j) \cup \{(z_g, z_i)\}$. Note that $T(z_i, z_j)$ is a left subtree of $P$. $G(z_i, z_j)$ is called a *left* subgraph of $G(w_h, w_{h+1})$.

If $\mu_{i+1}$ is the left child of $\mu_i$ (see Fig. 8(c)), then $G_i = G_{i-1} \cup G(z_j, z_i) \cup \{(z_g, z_i)\}$. Note that $T(z_j, z_i)$ is a right subtree of $P$. $G(z_j, z_i)$ is called a *right* subgraph of $G(w_h, w_{h+1})$.
- If $i = r$, then, $\mu_r$ is a leaf. $G_i = G_{i-1} \cup \{(z_j, z_i)\} \cup \{(z_g, z_i)\}$.

Let $L$ (respectively, $R$) be the set of all the left (respectively, right) subgraphs of $G(w_h, w_{h+1})$.
This completes the description of the structure of $G(s, t)$.

### 5.4. Child graphs of $G(s, t)$

The following are each called a *child* graph of $G(s, t)$: $G(s, u_1)$, each $G(u_i, u_{i+1})$, such that $i \neq b$, each $G(v_i, v_{i+1})$, such that $1 \leqslant i \leqslant c-1$, each $G(w_i, w_{i+1})$, such that $i \neq h$ and there is an edge $(w_i, w_{i+1})$ in $G(s, t)$, each $G(z_i, z_j) \in L$, and each $G(z_j, z_i) \in R$.

Fig. 9 shows the overall structure of $G(s, t)$ with respect to $P$ in the general case where $b \neq k$, $m \geqslant 2$, $1 \leqslant c \leqslant m-1$, and Case II applies to $G(u_b, u_{b+1})$. The structure of $G(s, t)$ in the special cases where $b = k$, or $m < 2$, or $c = 0$, or $c = m$, or Case I applies to $G(u_b, u_{b+1})$ have similar schematics.

## 6. Drawing a maximal outerplane graph $G(s, t)$

Let $\lambda$ be an integer such that $\lambda \in \{-1, 0, 1\}$. Let $\Gamma(s, t, \lambda)$ be a drawing of $G(s, t)$ where $y(t) - y(s) = \lambda$ and $x(t) - x(s) = |G(s, t)| - 1$, and that is defined recursively as follows (we will prove later that $\Gamma(s, t, \lambda)$ is a trapezoidal drawing):

If $G(s, t)$ consists of only one edge, namely, $(s, t)$, then $\Gamma(s, t, \lambda)$ consists of a single line-segment, $(s, t)$, such that $y(t) - y(s) = \lambda$, and $x(t) - x(s) = |G(s, t)| - 1 = 2 - 1 = 1$.

If $G(s, t)$ consists of only one internal face $f$, then $\Gamma(s, t, \lambda)$ consists of only one triangle with three vertices $s$, $t$, and $u_1$ such that $y(t) - y(s) = \lambda$, and $x(t) - x(s) = |G(s, t)| - 1 = 3 - 1 = 2$, $x(u_1) - x(s) = 1$, and $y(u_1) = y(s) + 1$. Note that $y(u_1) > \min\{y(s), y(t)\}$.

If $G(s, t)$ contains at least two internal faces, then let $P$ be a spine of the dual tree $T(s, t)$ of $G(s, t)$. We will use the notation of Section 5 in the rest of this section. Exactly one edge among $(s, u_1)$ and $(u_1, t)$ is the dual of an edge of $P$; assume that $(u_1, t)$ is that edge (we can define $\Gamma(s, t, \lambda)$ in a symmetrical fashion if $(s, u_1)$ is that edge. In fact, the structures of $\Gamma(s, t, \lambda)$ in the two cases where $(s, u_1)$ and $(u_1, t)$, respectively, are the duals of an edge of $P$, are mirror-images of each other).

Drawing $\Gamma(s, t, \lambda)$ consists of the line-segment $(s, t)$, drawing $\Gamma(s, u_1, 1)$, and a drawing $D(u_1, t, \sigma)$ of $G(u_1, t)$ that are arranged as in Fig. 10(a), such that

- $y(t) - y(s) = \lambda$, $x(t) - x(s) = |G(s, t)| - 1$, and
- $y(u_1) - y(s) = 1$, $x(u_1) - x(s) = |G(s, u_1)| - 1$.

Note that $x(t) - x(u_1) = x(t) - x(s) - (x(u_1) - x(s)) = |G(s, t)| - 1 - (|G(s, u_1)| - 1) = |G(s, t)| - |G(s, u_1)| = |G(u_1, t)| - 1$. Let $\sigma = y(u_1) - y(t) = 1 - \lambda$ be an integer. Note that $\sigma = 0$ if $\lambda = 1$, $\sigma = 1$ if $\lambda = 0$, and $\sigma = 2$ if $\lambda = -1$. Hence, $\sigma \in \{0, 1, 2\}$. Figs. 10(b),) (c), and (d) show $\Gamma(s, t, 1)$, $\Gamma(s, t, 0)$, and $\Gamma(s, t, -1)$, respectively.

As explained in Section 5, $G(u_1, t)$, and therefore $D(u_1, t, \sigma)$, contains at least one internal face.

### 6.1. Definition of $D(u_1, t, \sigma)$, where $\sigma \in \{0, 1, 2\}$

Drawing $D(u_1, t, \sigma)$, where $\sigma \in \{0, 1, 2\}$, is defined as follows:

Recall the definitions of integer $b$ from Section 5.1 and integer $c$ from Section 5.2. Also recall from Section 5.2 that $m$ is the number of neighbors of $u_b$ and $u_{b+1}$ in $G(u_b, u_{b+1})$ such that no neighbor is equal to $u_b$ or $u_{b+1}$. As explained in Section 5.2, if $1 \leqslant b \leqslant k-1$, then $m \geqslant 1$.

We have three cases: $b = 1$, $2 \leqslant b \leqslant k-1$, and $b = k$. Drawing $D(u_1, t, \sigma)$ is shown in these three cases in Figs. 11, 12, and 13, respectively.

- If $b = 1$, then: (see Fig. 11)
  - $D(u_1, t, \sigma)$ contains a drawing $D(u_1, u_2, 1)$ of $G(u_1, u_2)$, and the drawing $\Gamma(u_i, u_{i+1}, 0)$ for each $i$, where $2 \leqslant i \leqslant k-1$,

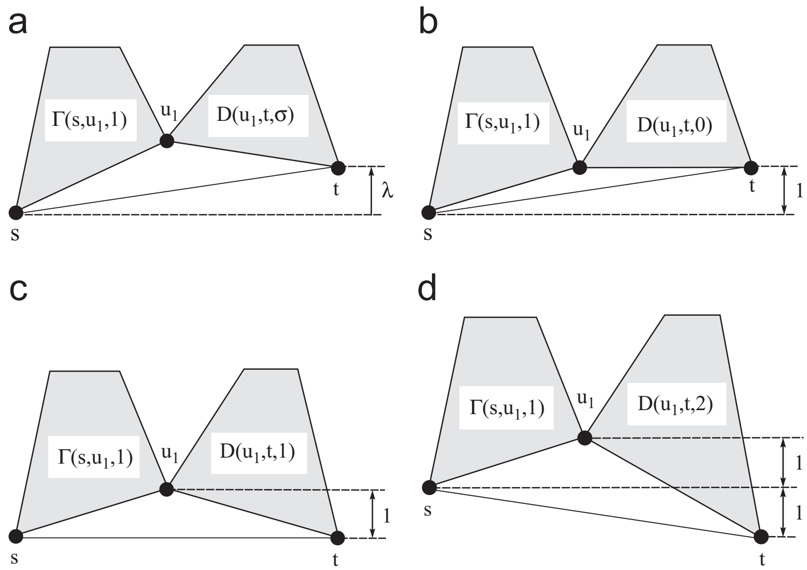


Fig. 10. Structure of $\Gamma(s,t,\lambda)$ if $G(s,t)$ contains at least two internal faces: (a) $\Gamma(s,t,\lambda)$, (b) $\Gamma(s,t,1)$, (c) $\Gamma(s,t,0)$, and (d) $\Gamma(s,t,-1)$.

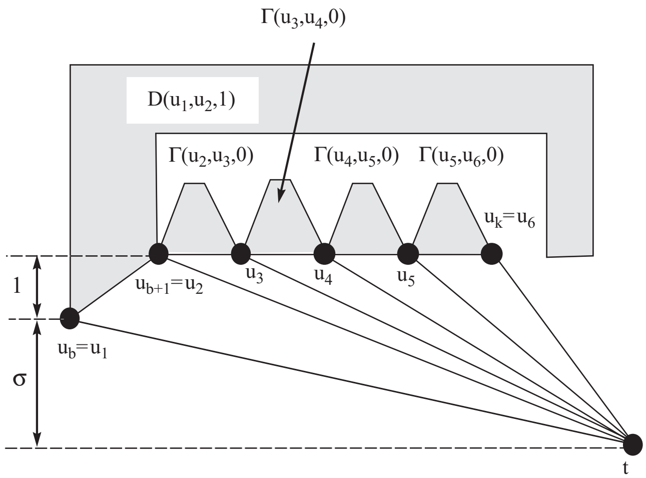


Fig. 11. Structure of $D(u_1,t,\sigma)$ if $b=1$.

- $y(u_1)-y(t)=\sigma$, $x(t)-x(u_1)=|G(u_1,t)|-1$,
- the value of $x(u_2)-x(u_1)$ is as follows: (note that since $b=1$, $x(u_{b+1})-x(u_b)=x(u_2)-x(u_1)$)
  if $m=1$, then $x(u_2)-x(u_1)=1$,
  otherwise, if $c=0$, then $x(u_2)-x(u_1)=m-1$,
  otherwise (i.e., if $m\geqslant 2$ and $c\geqslant 1$), then $x(u_2)-x(u_1)=m-c+\sum_{1\leqslant i\leqslant c-1}(|G(v_i,v_{i+1})|-1)$,
- $y(u_2)-y(u_1)=1$, and
- for each $i$, where $2\leqslant i\leqslant k-1$, $y(u_{i+1})=y(u_i)=y(u_2)$, and $x(u_{i+1})-x(u_i)=|G(u_i,u_{i+1})|-1$.

• If $2\leqslant b\leqslant k-1$, then: (see Fig. 12)
  - $D(u_1,t,\sigma)$ contains the drawing $\Gamma(u_1,u_2,1)$, a drawing $D(u_b,u_{b+1},0)$ of $G(u_b,u_{b+1})$, and the drawing $\Gamma(u_i,u_{i+1},0)$ for each $i$, where $i\neq b$ and $2\leqslant i\leqslant k-1$.
  - $y(u_1)-y(t)=\sigma$, and $x(t)-x(u_1)=|G(u_1,t)|-1$,
  - $y(u_2)-y(u_1)=1$, $x(u_2)-x(u_1)=|G(u_1,u_2)|-1$,

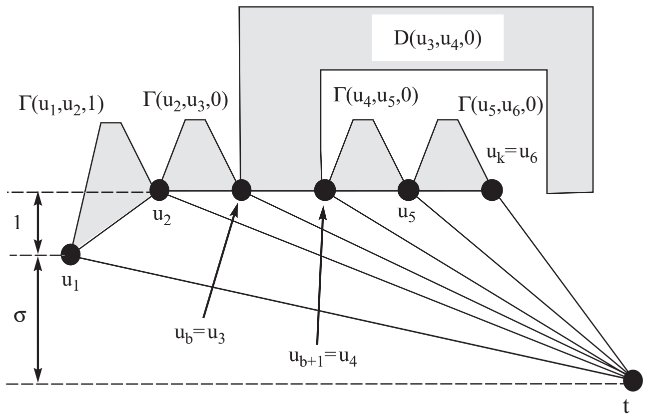


Fig. 12. Structure of $D(u_1, t, \sigma)$ if $2 \leqslant b \leqslant k-1$. Here, $b=3$.

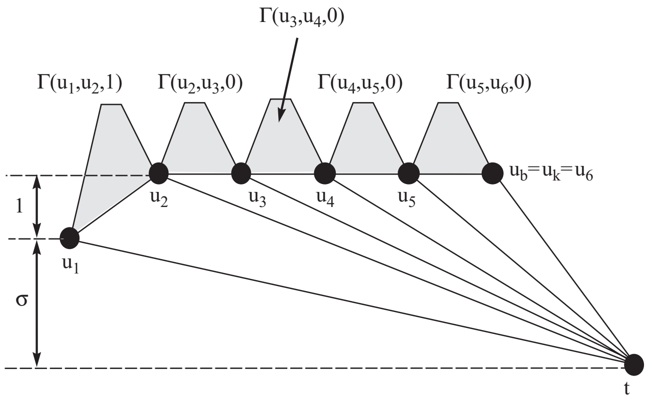


Fig. 13. Structure of $D(u_1, t, \sigma)$ if $b=k$.

- the value of $x(u_{b+1}) - x(u_b)$ is defined as follows:
  if $m=1$, then $x(u_{b+1}) - x(u_b) = 1$,
  otherwise, if $c=0$, then $x(u_{b+1}) - x(u_b) = m-1$,
  otherwise (i.e., if $m \geqslant 2$ and $c \geqslant 1$), then $x(u_{b+1}) - x(u_b) = m - c + \sum_{1 \leqslant i \leqslant c-1} (|G(v_i, v_{i+1})| - 1)$,
- $y(u_{b+1}) = y(u_b) = y(u_2)$, and
- for each $i$,where $i \neq b$ and $2 \leqslant i \leqslant k-1$, $y(u_{i+1}) = y(u_i) = y(u_2)$, and $x(u_{i+1}) - x(u_i) = |G(u_i, u_{i+1})| - 1$.

• If $b=k$, then: (see Fig. 13)
- $D(u_1, t, \sigma)$ contains the drawing $\Gamma(u_1, u_2, 1)$, and the drawing $\Gamma(u_i, u_{i+1}, 0)$ for each $i$, where $2 \leqslant i \leqslant k-1$.
- $y(u_1) - y(t) = \sigma$, and $x(t) - x(u_1) = |G(u_1, t)| - 1$,
- $y(u_2) - y(u_1) = 1$, $x(u_2) - x(u_1) = |G(u_1, u_2)| - 1$, and
- for each $i$, where $2 \leqslant i \leqslant k-1$, $y(u_{i+1}) = y(u_i) = y(u_2)$, and $x(u_{i+1}) - x(u_i) = |G(u_i, u_{i+1})| - 1$.

Thus, $D(u_1, t, \sigma)$ is essentially the same in all the three cases, except that if $b=1$, then it contains $D(u_b, u_{b+1}, 1)$, if $2 \leqslant b \leqslant k-1$, then it contains $\Gamma(u_1, u_2, 1)$ and $D(u_b, u_{b+1}, 0)$, and if $b=k$, then it contains $\Gamma(u_1, u_2, 1)$.

### *6.2. Definition of $D(u_b, u_{b+1}, \eta)$, where $1 \leqslant b \leqslant k-1$ and $\eta \in \{0, 1\}$*

Drawing $D(u_b, u_{b+1}, \eta)$, where $1 \leqslant b \leqslant k-1$ and $\eta \in \{0, 1\}$, is defined as follows:

Recall from Section 5.2 that $m$ is the number of neighbors of $u_b$ and $u_{b+1}$ in $G(u_b, u_{b+1})$, where no neighbor is equal to $u_b$ or $u_{b+1}$. As explained in Section 5.2, since $1 \leqslant b \leqslant k-1$, $m \geqslant 1$.

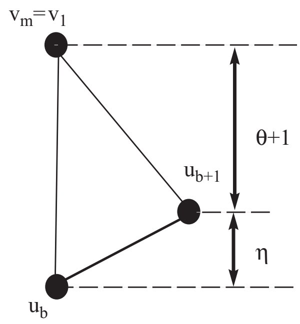


Fig. 14. Structure of $D(u_b, u_{b+1}, \eta)$, where $1 \leqslant b \leqslant k-1$ and $\eta \in \{0, 1\}$, with $m = 1$.

If $m = 1$, then $D(u_b, u_{b+1}, \eta)$ consists of exactly one triangle with vertices $u_b$, $u_{b+1}$, and $v_1$, such that $y(u_{b+1}) - y(u_b) = \eta$, $x(u_{b+1}) - x(u_b) = 1$, $y(v_1) - y(u_b) = \eta + \theta + 1$, and $x(v_1) = x(u_b)$, where $\theta = max_{b+1 \leqslant i \leqslant k-1} H(\Gamma(u_i, u_{i+1}, 0))$ (see Fig. 14).

Now assume that $m \geqslant 2$. Recall the definitions of integers $c$, and $h$ from Section 5.2. Also recall from Section 5.2 that $q$ is the number of neighbors of $v_c, v_{c+1}, \ldots, v_m$ in the graph $G' = G(v_c, v_{c+1}) \cup G(v_{c+1}, v_{c+2}) \cup \cdots \cup G(v_{m-1}, v_m)$, such that no neighbor is equal to any of $v_c, v_{c+1}, \ldots, v_m$.

We have three cases: $c = 0$, $c = m$, and $1 \leqslant c \leqslant m - 1$. Drawing $D(u_b, u_{b+1}, \eta)$ in these three cases is shown in Figs. 15, 16, and 17, respectively. Let $\theta$ be the integer that is defined as follows:

- if $q = 0$ then $\theta = \max_{b+1 \leqslant i \leqslant k-1} H(\Gamma(u_i, u_{i+1}, 0))$, and
- if $q \geqslant 1$, then $\theta = \max\{\max_{b+1 \leqslant i \leqslant k-1} H(\Gamma(u_i, u_{i+1}, 0)), \max_{h+1 \leqslant i \leqslant q-1} H(\Gamma(w_i, w_{i+1}, 0)), \max_{G(z_i, z_j) \in L} H(\Gamma(z_i, z_j, 0))\}$, where for convenience, we define $H(\Gamma(w_i, w_{i+1}, 0)) = 0$ if $G(w_i, w_{i+1})$ does not exist (which will happen if there is no edge $(w_i, w_{i+1})$).

(As we will show later, this value of $\theta$ helps in the planarity of $\Gamma(s, t, \lambda)$.)

In all the three cases,

- for each $i$, where $1 \leqslant i \leqslant c - 1$, $D(u_b, u_{b+1}, \eta)$ contains the drawing $\Gamma(v_i, v_{i+1}, 0)$,
- $y(u_{b+1}) - y(u_b) = \eta$, the value of $x(u_{b+1}) - x(u_b)$ is as given in Section 6.1 (note that if $b = 1$, then $x(u_{b+1}) - x(u_b) = x(u_2) - x(u_1)$),
- $y(v_1) - y(u_b) = \eta + \theta + 1 + \min\{m - 1, m - c\}$, $x(v_1) = x(u_b)$,
- $y(v_m) - y(u_{b+1}) = \theta + 1$, $x(v_m) = x(u_{b+1})$,
- for each $i$, where $1 \leqslant i \leqslant c - 1$, $y(v_{i+1}) = y(v_i) = y(v_1)$, and $x(v_{i+1}) - x(v_i) = |G(v_i, v_{i+1})| - 1$,
- for each $i$, where $c + 1 \leqslant i \leqslant m - 1$, $y(v_i) - y(v_{i+1}) = 1$, and $x(v_{i+1}) - x(v_i) = 1$.

If $c = m$, then we are done with the definition of $D(u_b, u_{b+1}, \eta)$ (see Fig. 16).

If $c = 0$, then there are two possibilities: $q = 0$ and $q \geqslant 1$.

If $1 \leqslant c \leqslant m - 1$, then there is only one possibility: $q \geqslant 1$. (If $1 \leqslant c \leqslant m - 1$, then $G'$ has at least one internal face, which corresponds to the vertex of $P$ that is an end-point of the dual of $(v_c, v_{c+1})$. So, $q = 0$ is not possible.)

If $q = 0$, then also we are done with the definition of $D(u_b, u_{b+1}, \eta)$ (see Fig. 15(a)).

Now assume that $q \geqslant 1$ (see Figs. 15(b) and 17(a,b)). Let $\chi$ be the integer, such that $\chi = \sum_{b+1 \leqslant i \leqslant k-1} (|G(u_i, u_{i+1})| - 1) + \max\{\sum_{1 \leqslant i \leqslant h-1} \max\{|G(w_i, w_{i+1})| - 1, 1\}, \sum_{h+1 \leqslant i \leqslant q-1} \max\{|G(w_i, w_{i+1})| - 1, 1\}\}$, where for convenience, we define $|G(w_i, w_{i+1})| = 0$ if $G(w_i, w_{i+1})$ does not exist (which will happen if there is no edge $(w_i, w_{i+1})$).

Recall from Section 5.2 that there can be two cases, I and II, depending upon whether an edge among $(w_1, w_2)$, $(w_2, w_3), \ldots, (w_{q-1}, w_q)$ is the dual of an edge of $P$. In both Cases I and II (see Figs. 15(b) and 17(a,b))),

- $D(u_b, u_{b+1}, \eta)$ contains $\Gamma(w_i, w_{i+1}, 0)$ for each $i$, where $1 \leqslant i \leqslant h - 1$,
- $D(u_b, u_{b+1}, \eta)$ contains $\Gamma(w_i, w_{i+1}, 0)$ flipped left-to-right and upside-down, for each $i$, where $h + 1 \leqslant i \leqslant q - 1$,
- if $h \neq q$, then $y(w_{h+1}) = y(v_m) - 1$ and $x(w_{h+1}) - x(v_m) = x(w_{h+1}) - x(u_{b+1}) = \chi + 1$,
- if $h \neq 0$, then $y(w_h) = y(v_1) + 1$, $x(w_h) - x(v_m) = x(w_h) - x(u_{b+1}) = \chi + 1$,

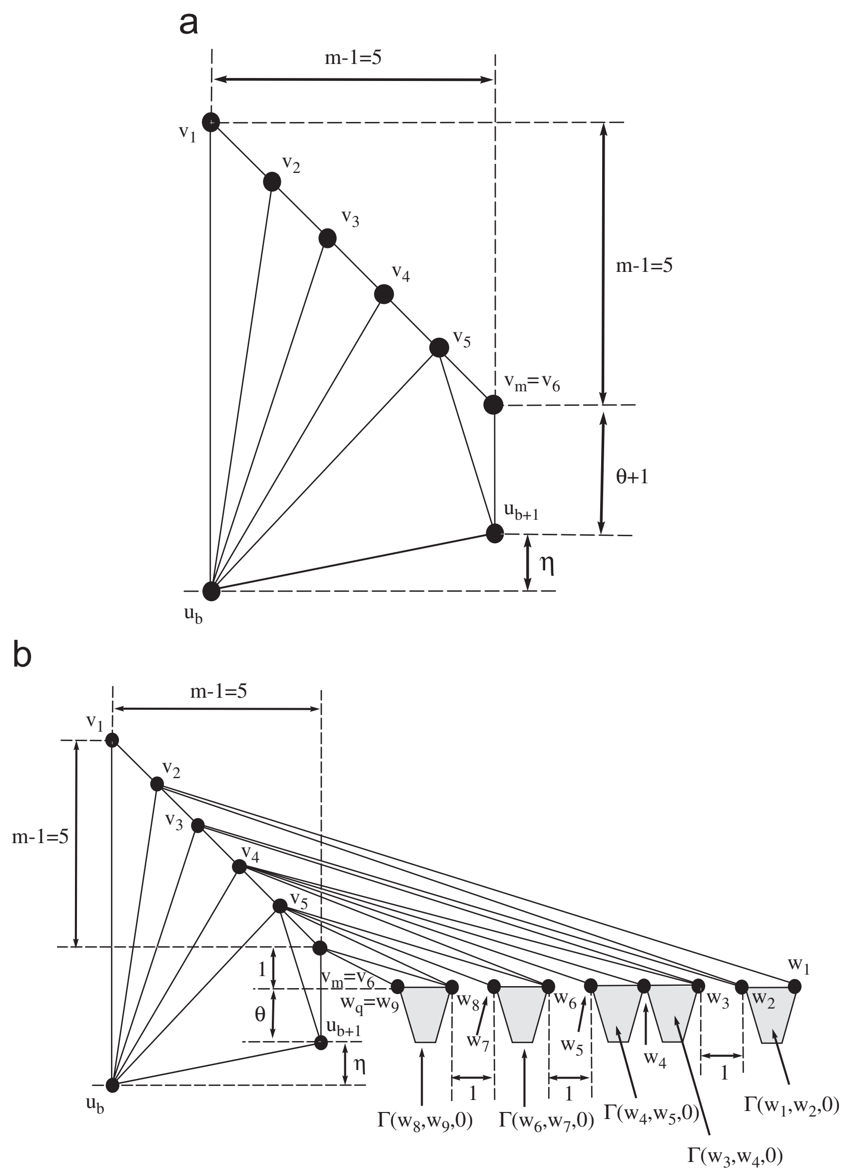


Fig. 15. Structure of a $D(u_b, u_{b+1}, \eta)$, where $1 \leqslant b \leqslant k-1$, $\eta \in \{0, 1\}$, $m \geqslant 2$, and $c = 0$, (a) if $q = 0$, and (b) if $q \geqslant 1$. Here, $m = 6$. In part (b), $q = 9$.

- for each $i$, where $1 \leqslant i \leqslant h-1$, $y(w_i) = y(v_1) + 1$, and $x(w_{i+1}) - x(w_i) = \max\{|G(w_i, w_{i+1})| - 1, 1\}$ where $|G(w_i, w_{i+1})| = 0$ if $G(w_i, w_{i+1})$ does not exist (which will happen if there is no edge $(w_i, w_{i+1})$), and
- for each $i$, where $h+1 \leqslant i \leqslant q-1$, $y(w_i) = y(v_m) - 1$, and $x(w_i) - x(w_{i+1}) = \max\{|G(w_i, w_{i+1})| - 1, 1\}$ where $|G(w_i, w_{i+1})| = 0$ if $G(w_i, w_{i+1})$ does not exist (which will happen if there is no edge $(w_i, w_{i+1})$).

Moreover, in Case II only, $D(u_b, u_{b+1}, \eta)$ contains a drawing $D(w_h, w_{h+1})$ of $G(w_h, w_{h+1})$ (in Case I, there is no edge $(w_h, w_{h+1})$).

If Case I applies to $G(u_b, u_{b+1})$, then we are done. Otherwise (i.e., if Case II applies), $D(w_h, w_{h+1})$ is as defined in Section 6.3. Note that in Case II, $y(w_h) - y(w_{h+1}) = y(v_1) + 1 - (y(v_m) - 1) = y(v_1) - y(v_m) + 2 = m - c + 2$.

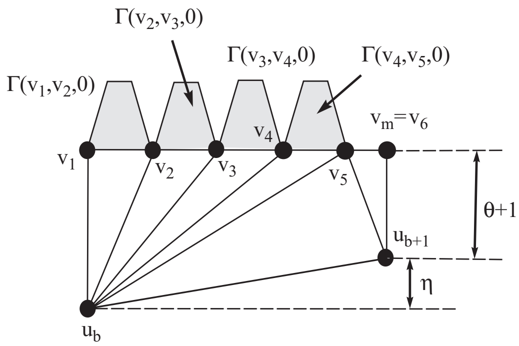


Fig. 16. Structure of a $D(u_b, u_{b+1}, \eta)$, where $1 \leqslant b \leqslant k-1$, $\eta \in \{0, 1\}$, $m \geqslant 2$, and $c = m$. Here, $m = 6$.

### 6.3. Definition of $D(w_h, w_{h+1})$, where $(w_h, w_{h+1})$ is the dual of an edge of $P$

Recall the definitions of the subpath $M = \mu_1\mu_2 \ldots \mu_r$ of $P$, graphs $G_0, G_1, G_2, \ldots, G_r$, and sets $L$ and $R$ from Section 5.3.

Drawing $D(w_h, w_{h+1})$ is defined recursively as a sequence of drawings, $D_0, D_1, \ldots, D_r$, where $D(w_h, w_{h+1}) = D_r$, and for each $i$, where $0 \leqslant i \leqslant r$, $D_i$ is a drawing of $G_i$.

- Drawing $D_0$ consists of a single line-segment $(w_h, w_{h+1})$, such that $y(w_h) - y(w_{h+1}) = m - c + 2$ and $x(w_h) = x(w_{h+1})$.
- For each $i$, where $1 \leqslant i \leqslant r-1$, $D_i$ is constructed from $D_{i-1}$ as shown in Fig. 18(a,b).
  - if $\mu_{i+1}$ is the right child of $\mu_i$, then $D_i$ is as shown in Fig. 18(a).
    Drawing $D_i$ consists of $D_{i-1}$, line-segment $(z_g, z_i)$, and $\Gamma(z_i, z_j, 0)$ flipped left-to-right and upside-down, and $y(z_i) = y(z_j) = y(w_{h+1})$, and $x(z_i) - x(z_j) = |G(z_i, z_j)| - 1$.
  - if $\mu_{i+1}$ is the left child of $\mu_i$, then $D_i$ is as shown in Fig. 18(b).
    Drawing $D_i$ consists of $D_{i-1}$, line-segment $(z_g, z_i)$, and $\Gamma(z_j, z_i, 0)$, and
    $y(z_i) = y(z_j) = y(w_h)$, and $x(z_i) - x(z_j) = |G(z_j, z_i)| - 1$.
- If $i = r$, then assume without any loss of generality that $y(z_j) < y(z_g)$. Drawing $D_i$ is constructed from $D_{i-1}$ as shown in Fig. 18(c), such that $y(z_r) = y(z_j)$ and $x(z_r) = \max\{x(z_j), x(z_g)\} + 1$.

Fig. 19 shows the general structure of $D(w_h, w_{h+1})$.
This completes the definition of $\Gamma(s, t, \lambda)$.

### 6.4. Child drawings of $\Gamma(s, t, \lambda)$

The following are each called a *child* drawing of $\Gamma(s, t, \lambda)$: $\Gamma(s, u_1, 1)$, $\Gamma(u_1, u_2, 1)$ if $b \neq 1$, each $\Gamma(u_i, u_{i+1}, 0)$ such that $i \neq b$, each $\Gamma(v_i, v_{i+1}, 0)$ such that $1 \leqslant i \leqslant c-1$, each $\Gamma(w_i, w_{i+1}, 0)$ such that $i \neq h$ and there is an edge $(w_i, w_{i+1})$ in $G(s, t)$, each $\Gamma(z_i, z_j, 0)$ such that $G(z_i, z_j) \in L$, and each $\Gamma(z_j, z_i, 0)$ such that $G(z_j, z_i) \in R$.

Figs. 20, 21, and 22 show the overall structures of $\Gamma(s, t, 1)$, $\Gamma(s, t, 0)$, and $\Gamma(s, t, -1)$, respectively, in terms of their child drawings in the general case where $2 \leqslant b \leqslant k-1$, $m \geqslant 2$, $1 \leqslant c \leqslant m-1$, and Case II applies to $G(u_b, u_{b+1})$. The structures of $\Gamma(s, t, 1)$, $\Gamma(s, t, 0)$, and $\Gamma(s, t, -1)$ in the special cases where $b = 1$, or $b = k$, or $m < 2$, or $c = 0$, or $c = m$, or Case I applies to $G(u_b, u_{b+1})$ have similar schematics. In each case, $\Gamma(s, t, 1)$, $\Gamma(s, t, 0)$, and $\Gamma(s, t, -1)$ are identical except for the relative positions of $s$ and $t$ (see for example, Figs. 20–22).

### 6.5. Technical lemmas related to $\Gamma(s, t, \lambda)$

**Lemma 5** (*planarity*). *Drawing $\Gamma(s, t, \lambda)$ is trapezoidal.*

**Proof.** We will prove using induction that $\Gamma(s, t, \lambda)$ is a trapezoidal drawing.

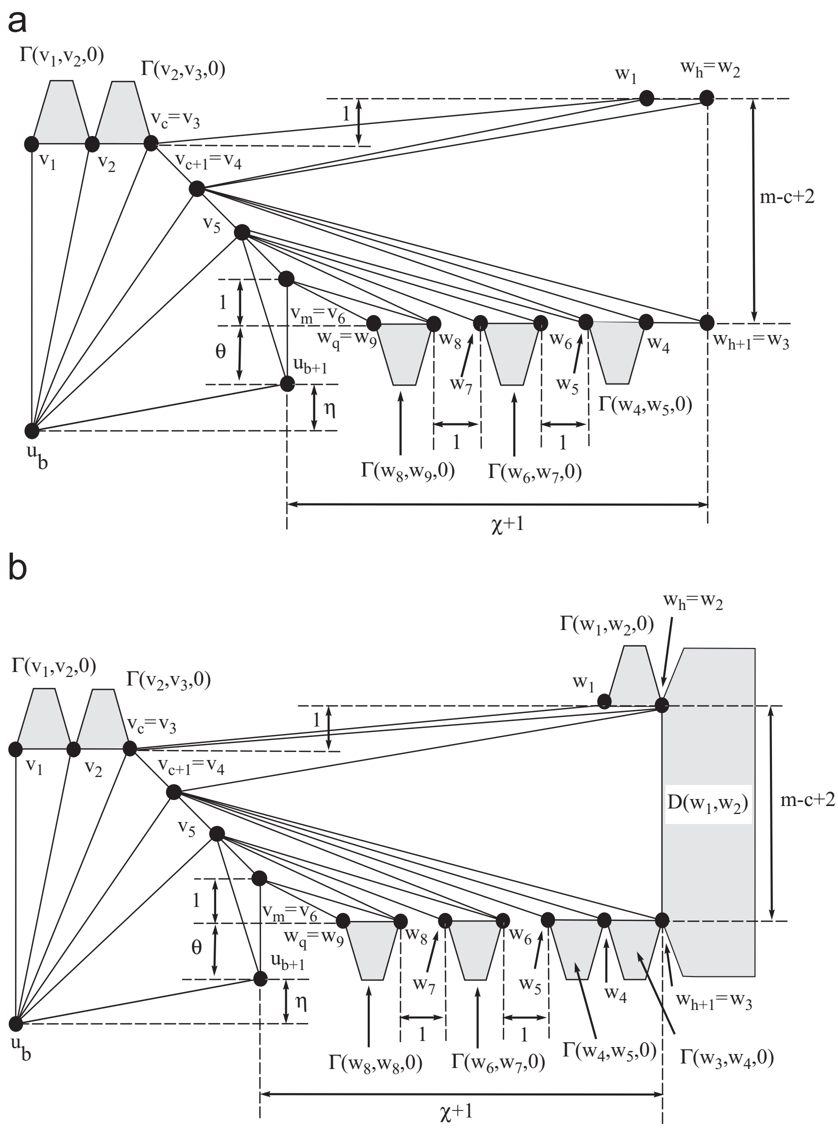


Fig. 17. Structure of a $D(u_b, u_{b+1}, \eta)$, where $1 \leqslant b \leqslant k-1$, $\eta \in \{0, 1\}$, $m \geqslant 2$ and $1 \leqslant c \leqslant m-1$, (a) in Case I (with $h=2$), and (b) in Case II. Here, $c=3$, $h=2$, $m=6$, and $q=9$.

If $G(s,t)$ consists of only one edge (namely, $(s,t)$), or of only one internal face, then trivially $\Gamma(s,t,\lambda)$ is a trapezoidal drawing.

If $G(s,t)$ contains at least two internal faces, then we give the proof for the case where $(u_1, t)$ is the dual of an edge of $P$ (the proof for the case where $(s, u_1)$ is the dual of an edge of $P$ is similar).

We give the proof for the general case where $2 \leqslant b \leqslant k-1$, $m \geqslant 2$, $1 \leqslant c \leqslant m-1$, and Case II applies to $G(u_b, u_{b+1})$. The proofs for the special cases where $b=1$, or $b=k$, or $m<2$, or $c=0$, or $c=m$, or Case I applies to $G(u_b, u_{b+1})$ are similar.

We give the proof for $\Gamma(s,t,1)$. The proofs for $\Gamma(s,t,0)$ and $\Gamma(s,t,-1)$ are similar.

See Fig. 20. From the inductive hypothesis, each child drawing of $\Gamma(s,t,1)$ is a trapezoidal drawing.

We will first prove that for each vertex $o$ of $G(s,t)$ that is different from $s$ and $t$, $x(s) < x(o) < x(t)$, and $y(o) > \min\{y(s), y(t)\}$. Then, we will prove that $\Gamma(s,t,1)$ is planar. This will prove that $\Gamma(s,t,1)$ is a trapezoidal drawing.

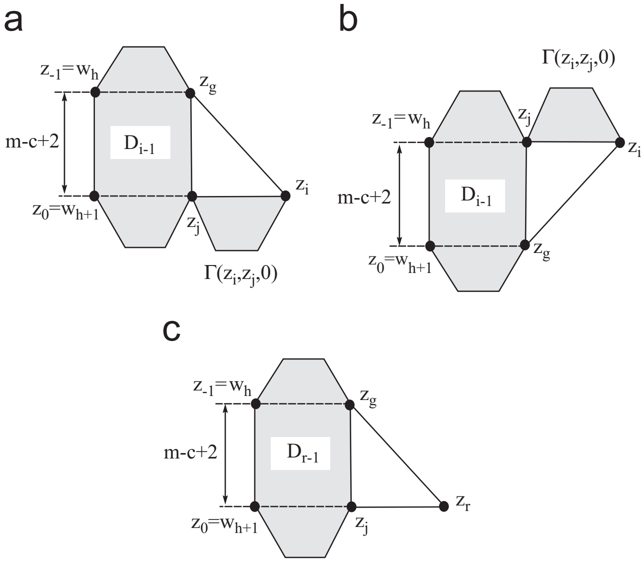


Fig. 18. Constructing $D_i$ from $D_{i-1}$ (a) if $\mu_{i+1}$ is the right child of $\mu_i$, and (b) if $\mu_{i+1}$ is the left child of $\mu_i$, and (c) if $i = r$. In part (c), for simplicity, we have placed $z_g$ directly above $z_j$, but in general $x(z_g) \neq x(z_j)$.

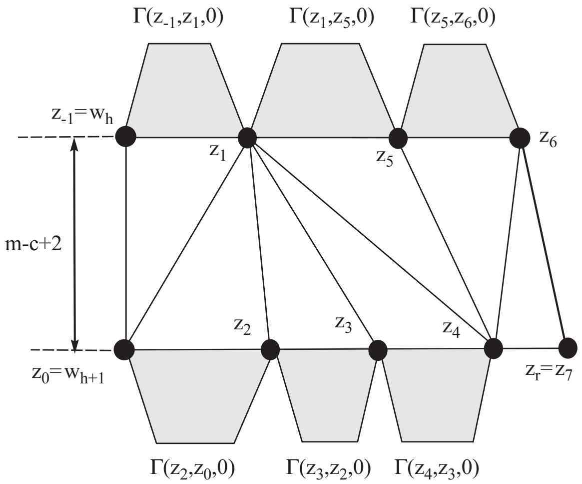


Fig. 19. Structure of a $D(w_h, w_{h+1})$, where $(w_h, w_{h+1})$ is the dual of an edge of $P$. Here, $r = 7$.

We have that $x(u_k) - x(u_{b+1}) = \sum_{b+1 \leqslant i \leqslant k-1} (|G(u_i, u_{i+1})| - 1)$. Hence, $x(w_{h+1}) - x(u_k) = x(w_{h+1}) - x(u_{b+1}) - (x(u_k) - x(u_{b+1})) = \chi + 1 - \sum_{b+1 \leqslant i \leqslant k-1} (|G(u_i, u_{i+1})| - 1) = \max\{\sum_{1 \leqslant i \leqslant h-1} \max\{|G(w_i, w_{i+1})| - 1, 1\}, \sum_{h+1 \leqslant i \leqslant q-1} \max\{|G(w_i, w_{i+1})| - 1, 1\}\} + 1 = \max\{x(w_{h+1}) - x(w_1), x(w_{h+1}) - x(w_q)\} + 1 = x(w_{h+1}) - \min\{x(w_1), x(w_q)\} + 1$. Hence, $x(u_k) = \min\{x(w_1), x(w_q)\} - 1$.

Hence, $x(v_r) > x(w_{h+1}) \geqslant \min\{x(w_1), x(w_q)\} = x(u_k) + 1 > x(u_k)$.

Let $G' = G(v_c, v_{c+1}) \cup G(v_{c+1}, v_{c+2}) \cup \cdots \cup G(v_{m-1}, v_m)$. Vertex $u_k$ has the greatest $x$-coordinate among all the vertices of the graph $G(s,t) - G'$, except $t$. Vertex $v_r$ has the greatest $x$-coordinate among all the vertices of $G'$. Since $x(v_r) > x(u_k)$, it follows that among all the vertices of $G(s,t)$ that are different from $t$, $v_r$ has the greatest $x$-coordinate. It is also easy to see that among all the vertices of $G(s,t)$ that are different from $s$, the vertex with the least $x$-coordinate is in $G(s, u_1)$, and its $x$-coordinate is greater than that of $s$. We will prove that $x(v_r) < x(t)$, and this will prove that for each vertex $o$ of $G(s,t)$ that is different from $s$ and $t$, $x(s) < x(o) < x(t)$.

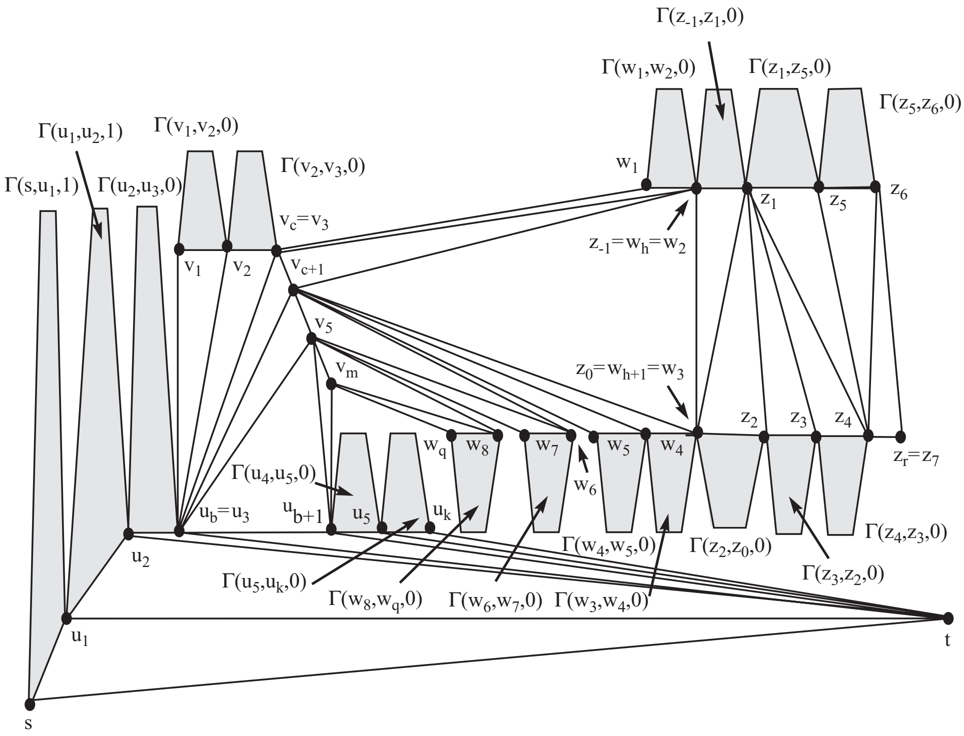


Fig. 20. Structure of a $\Gamma(s,t,1)$ in terms of its child drawings in the general case where $2 \leqslant b \leqslant k-1$, $m \geqslant 2$, $1 \leqslant c \leqslant m-1$, and Case II applies to $G(u_b,u_{b+1})$. Note that here $u_b=u_3$, $u_{b+1}=u_4$, $u_k=u_6$, $v_c=v_3$, $v_{c+1}=v_4$, $v_m=v_6$, $z_{-1}=w_h=w_2$, $z_0=w_{h+1}=w_3$, $w_q=w_9$, and $z_r=z_7$.

Assume the plane is covered by an infinite grid consisting of horizontal rows and vertical columns, where each column (respectively, row) contains the points with the same integer $x$-coordinate (respectively, $y$-coordinate), and the horizontal (respectively, vertical) distance between adjacent columns (respectively, rows) is equal to 1.

Let $C$ be the set of all the child graphs of $G(s,t)$. A vertex can be in at most two child graphs; if it is in two child graphs, then it is the common end-point of their respective reference edges. Let $A$ be the set of all the vertices that are in two child graphs. Let $B=\{v_i | c+1 \leqslant i \leqslant m\}$. Note that $|B|=m-c$.

We have proved earlier that $x(u_k)=\min\{x(w_1),x(w_q)\}-1$. Also, for each $i$, such that $i \neq h$, $1 \leqslant i \leqslant q-1$, and there is no edge $(w_i,w_{i+1})$, we have that $|x(w_{i+1})-x(w_i)|=\max\{|G(w_i,w_{i+1})|-1,1\}=\max\{0-1,1\}=1$. Hence, it can be easily verified that each column that intersects the line-segment $(s,v_r)$ either contains a $v_i$, where $c+1 \leqslant i \leqslant m$, or intersects the drawing of a child graph (see Fig. 20). The drawing $\Gamma''$ of each child graph $G''$ is trapezoidal. Hence, the width of $\Gamma''$ is equal to $|G''|-1$. Hence, the total number of columns intersecting $\Gamma''$ is equal to $|G''|$. Hence, the total number of columns intersecting the line-segment $(s,v_r)$ is at most $|B|+\sum_{G''\in C}|G''|-|A|$, where the term $-|A|$ appears because the column containing each vertex of $A$ is counted twice in $\sum_{G''\in C}|G''|$.We also have that $\sum_{G''\in C}|G''|=|G(s,t)|-1-|B|+|A|$, where the term $-1-|B|$ appears because $t$ and the vertices of set $B$ are not in any child graph, and the term $+|A|$ appears because each vertex of $A$ is counted twice in $\sum_{G''\in C}|G''|$. Hence, the total number of columns intersecting $(s,v_r)$ is at most $|B|+|G(s,t)|-1-|B|+|A|-|A|=|G(s,t)|-1$. Since $x(v_r)-x(s)$ is one less than the number of columns intersecting $(s,v_r)$, $x(v_r)-x(s) \leqslant |G(s,t)|-1-1 < |G(s,t)|-1=x(t)-x(s)$. Hence, $x(v_r)<x(t)$.

Now consider the $y$-coordinate of a vertex $o$ of $G(s,t)$ that is different from $s$ and $t$. First of all, notice that $y(u_b)>y(s)=\min\{y(s),y(t)\}$ (see Fig. 20).

Recall from Section 5.2 that $\theta=\max\{\max_{b+1\leqslant i\leqslant k-1} H(\Gamma(u_i,u_{i+1},0)), \max_{h+1\leqslant i\leqslant q-1} H(\Gamma(w_i,w_{i+1},0)), \max_{G(z_i,z_j)\in L} H(\Gamma(z_i,z_j,0))\}$. Also, we have that $y(v_m)-y(u_{b+1})=\theta+1$, $y(w_i)=y(v_m)-1$ for each $i$, where $h+1 \leqslant i \leqslant q-1$, and $y(z_i)=y(w_{h+1})=y(v_m)-1$ for each $i$, such that $G(z_i,z_j)\in L$. Hence, if $o\in G(w_i,w_{i+1})$, where $h+1 \leqslant i \leqslant q-1$, then $y(o) \geqslant y(w_i)-H(\Gamma(w_i,w_{i+1},0)) \geqslant y(w_i)-\theta=y(v_m)-1-\theta=y(u_{b+1})=y(u_b)>\min\{y(s),y(t)\}$.

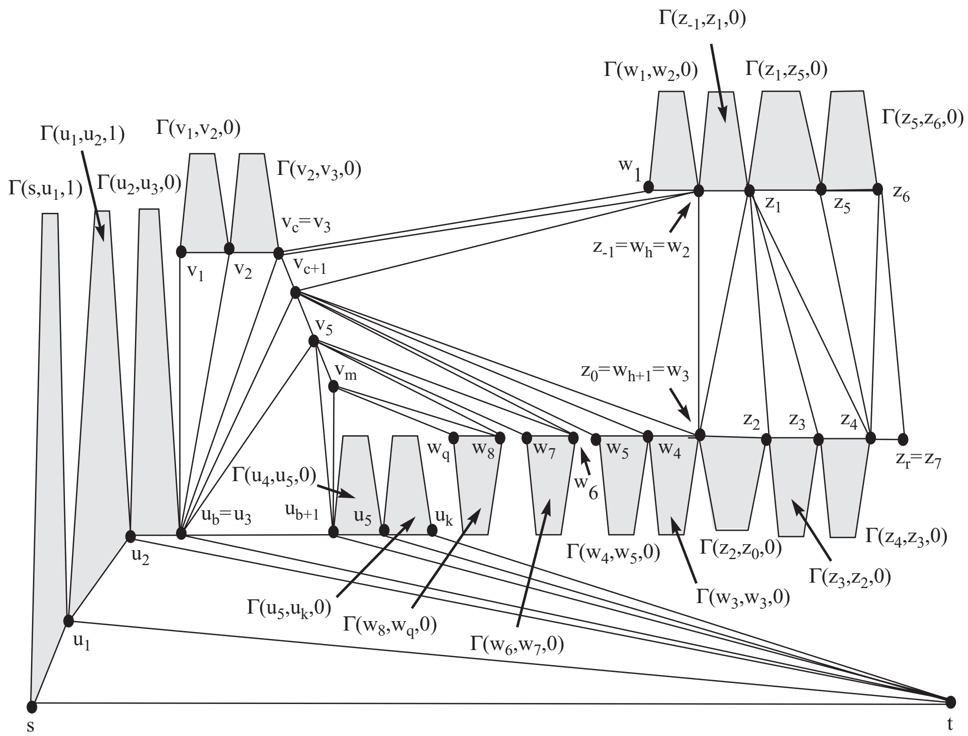


Fig. 21. Structure of a $\Gamma(s,t,0)$ in terms of its child drawings in the general case where $2\leqslant b\leqslant k-1$, $m\geqslant 2$, $1\leqslant c\leqslant m-1$, and Case II applies to $G(u_b,u_{b+1})$. Note that here $u_b=u_3$, $u_{b+1}=u_4$, $u_k=u_6$, $v_c=v_3$, $v_{c+1}=v_4$, $v_m=v_6$, $z_{-1}=w_h=w_2$, $z_0=w_{h+1}=w_3$, $w_q=w_9$, and $z_r=z_7$.

If $o\in G(z_i,z_j)$, where $G(z_i,z_j)\in L$, then $o\geqslant y(z_i)-H(\Gamma(z_i,z_j,0))\geqslant y(z_i)-\theta=y(v_m)-1-\theta=y(u_{b+1})=y(u_b)>\min\{y(s),y(t)\}$.

If $o\in G(s,u_1)$, then $y(o)>\min\{y(s),y(u_1)\}=y(s)=\min\{y(s),y(t)\}$ because $\Gamma(s,u_1,1)$ is a trapezoidal drawing where $\min\{y(s),y(u_1)\}=y(s)$.

If $o$ is in any other child graph of $G(s,t)$, then a similar reasoning shows that $y(o)\geqslant y(u_{b+1})=y(u_b)>\min\{y(s),y(t)\}$ because the drawing of each child graph is a trapezoidal drawing, and the end-points of the reference edges of these child graphs have $y$-coordinates greater than or equal to $y(u_b)$.

Finally, if $o=v_i$, where $c+1\leqslant i\leqslant m$, then also $y(o)\geqslant y(v_m)>y(u_{b+1})=y(u_b)>\min\{y(s),y(t)\}$.

We therefore conclude that $y(o)>\min\{y(s),y(t)\}$.

Finally, we will prove that $\Gamma(s,t,1)$ is planar. This will conclude the proof that $\Gamma(s,t,1)$ is trapezoidal.

Each child drawing is trapezoidal. From Lemma 3 and the relative placements of the child drawings in $\Gamma(s,t,1)$, no two child drawings overlap, except at the common end-point of their reference edges.

Let $o$ be a vertex such that $o\in G(w_i,w_{i+1})$, where $h+1\leqslant i\leqslant q-1$. As proved earlier, $y(o)\geqslant y(u_{b+1})=y(u_k)$. Also, $x(o)\geqslant x(w_q)\geqslant\min\{x(w_1),x(w_q)\}=x(u_k)+1>x(u_k)$. Since $y(t)<y(u_k)\leqslant y(o)$, and for each $i'$, where $1\leqslant i'\leqslant k$, $x(u_{i'})\leqslant x(u_k)<x(o)$, it follows that there is no crossing between an edge $(t,u_{i'})$, where $1\leqslant i'\leqslant k$, and an edge connecting two vertices of $G(w_i,w_{i+1})$.

Using a similar reasoning we can show that there is no crossing between any edge $(t,u_{i'})$, where $1\leqslant i'\leqslant k$, and any edge connecting two vertices of a $G(z_i,z_j)$, where $G(z_i,z_j)\in L$.

Now suppose $o\in G(u_i,u_{i+1})$, where $b+1\leqslant i\leqslant k-1$, then, $y(o)\leqslant y(u_i)+H(\Gamma(u_i,u_{i+1},0))=y(u_{b+1})+H(\Gamma(u_i,u_{i+1}),0)\leqslant y(u_{b+1})+\theta=y(v_m)-1$. For any $i'$, where $h+1\leqslant i'\leqslant q$, $y(w_{i'})=y(v_m)-1\geqslant y(o)$, and $x(w_{i'})\geqslant x(w_q)\geqslant\min\{w_1,w_q\}=x(u_k)+1>x(o)$. For any $i''$, where $c\leqslant i''\leqslant m$, $y(v_{i''})\geqslant y(v_m)>y(o)$. Hence, it follows that there is no crossing between an edge connecting two vertices of $G(u_i,u_{i+1})$, and an edge that connects $w_{i'}$ with a vertex among $v_c,v_{c+1},\ldots,v_m$.

It is easy to verify from Fig. 20 that there is no crossing between any other pair of edges. We therefore conclude that $\Gamma(s,t,1)$ is planar. $\square$

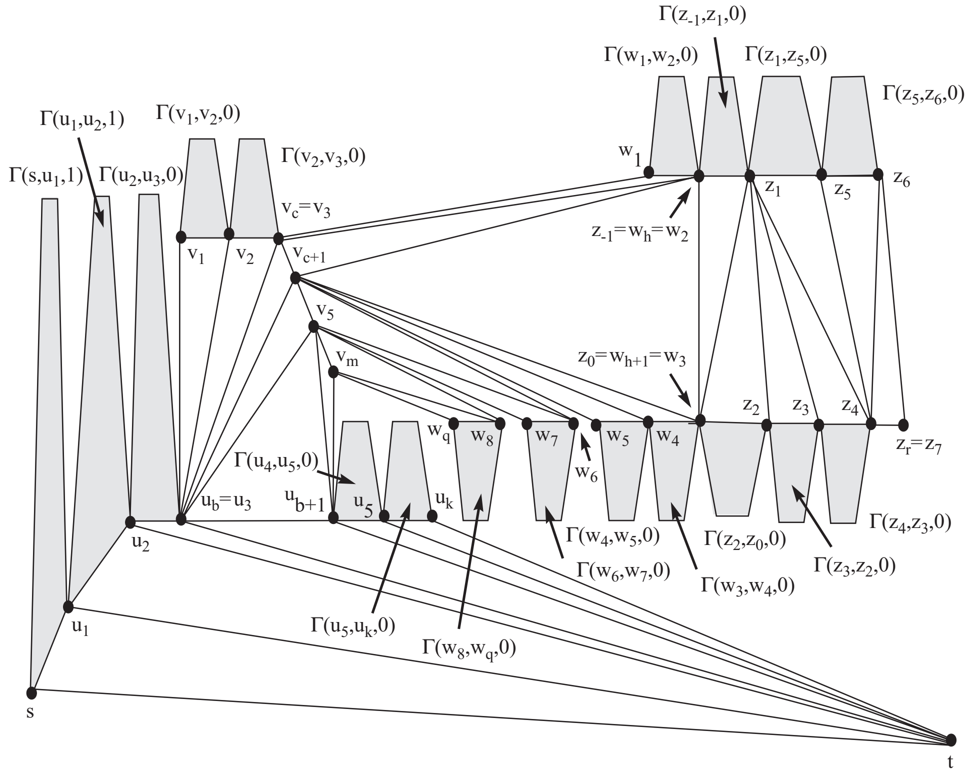


Fig. 22. Structure of a $\Gamma(s,t,-1)$ in terms of its child drawings in the general case where $2 \leqslant b \leqslant k-1$, $m \geqslant 2$, $1 \leqslant c \leqslant m-1$, and Case II applies to $G(u_b, u_{b+1})$. Note that here $u_b = u_3$, $u_{b+1} = u_4$, $u_k = u_6$, $v_c = v_3$, $v_{c+1} = v_4$, $v_m = v_6$, $z_{-1} = w_h = w_2$, $z_0 = w_{h+1} = w_3$, $w_q = w_9$, and $z_r = z_7$.

**Lemma 6** (*Time*). *Drawing $\Gamma(s,t,\lambda)$ can be constructed in* $O(|G(s,t)|)$ *time, assuming that we have already constructed $T(s,t)$, and computed the size of the subtree rooted at each vertex $\tau$ of $T(s,t)$, and stored it in $\tau$.*

**Proof.** If $G(s,t)$ consists of only one edge (namely, $(s,t)$), or of only one internal face, then $\Gamma(s,t,\lambda)$ can be trivially constructed in $O|G(s,t)|$ time.

If $G(s,t)$ contains at least two internal faces, then the definition of $\Gamma(s,t,\lambda)$ given above implicitly defines a recursive procedure for constructing it, where we first compute a spine $P$ of $G(s,t)$, then determine the child graphs of $G(s,t)$, then recursively construct the drawings of the child graphs, and finally combine the drawings to obtain $\Gamma(s,t,\lambda)$.

Let $E(s,t)$ be the set of the edges of $G(s,t)$. Let $E_N(s,t)$ be the set of the edges of $G(s,t)$ that are not in any child graph of $G(s,t)$. Let $E_R(s,t)$ be the set of the reference edges of the child graphs of $G(s,t)$. Let $C$ be the set of the child graphs of $G(s,t)$. Let $E_{TR}(s,t)$ be a set of the edges of $G(s,t)$ that is defined (recursively) as follows:

$$E_{TR}(s,t) = \bigcup_{G(s',t') \in C} E_{TR}(s',t') \cup E_R(s,t),$$

i.e., for each edge $e \in E_{TR}(s,t)$, either $e$ is the reference edge of a child graph of $G(s,t)$ or $e \in E_{TR}(s',t')$ for some child graph $G(s',t')$ of $G(s,t)$.

If $e \in E_R(s,t)$, then $e$ is not in $\bigcup_{G(s',t') \in C} E_{TR}(s',t')$. Hence, $|E_{TR}(s,t)| = |E_R(s,t)| + \sum_{G(s',t') \in C} |E_{TR}(s',t')|$.

We will prove using induction that we can construct $\Gamma(s,t,\lambda)$ in $O(|E(s,t)| + |E_{TR}(s,t)|)$ time.

Using Lemma 1, we can compute a spine $P$ of $G(s,t)$ in $O(|P|) = O(|E_N(s,t)| + |E_R(s,t)|)$ time. Determining the child graphs of $G(s,t)$ takes $O(|E_N(s,t)| + |E_R(s,t)|)$ time. By inductive hypothesis, constructing the drawing of a child graph $G(s',t')$ takes $O(|E(s',t')| + |E_{TR}(s',t')|)$ time. Combining the drawings of the child graphs to obtain $\Gamma(s,t,\lambda)$ takes $O(|E_N(s,t)| + |E_R(s,t)|)$ time.

Hence, the total time taken to construct $\Gamma(s,t,\lambda)$ is $O(|E_N(s,t)| + |E_R(s,t)|) + O(|E_N(s,t)| + |E_R(s,t)|) + \sum_{G(s',t')\in C} O(|E(s',t')| + |E_{TR}(s',t')|) + O(|E_N(s,t)| + |E_R(s,t)|) = O(|E_N(s,t)| + |E_R(s,t)|) + \sum_{G(s',t')\in C} O(|E(s',t')| + |E_{TR}(s',t')|) = O(|E_N(s,t)| + \sum_{G(s',t')\in C} |E(s',t')|) + O(|E_R(s,t)| + \sum_{G(s',t')\in C} |E_{TR}(s',t')|)$ time. Since, $|E_N(s,t)| + \sum_{G(s',t')\in C} |E(s',t')| = |E(s,t)|$, and $|E_R(s,t)| + \sum_{G(s',t')\in C} |E_R(s',t')|) = |E_{TR}(s,t)|$, the total time taken to construct $\Gamma(s,t,\lambda)$ is $O(|E(s,t)|) + O(|E_{TR}(s,t)|)$.

Since $|E(s,t)| + |E_{TR}(s,t)| = O(|G(s,t)|)$, it follows that we can construct $\Gamma(s,t,\lambda)$ in $O(|G(s,t)|)$ time. □

**Lemma 7** (*Area*). *Drawing $\Gamma(s,t,\lambda)$ has $O(d \cdot |G(s,t)|^{1+p})$ area, where $p = 0.48$, and $d$ is the degree of $G(s,t)$.*

**Proof.** If $G(s,t)$ consists of only one edge (namely, $(s,t)$), or of only one internal face, then trivially $\Gamma(s,t,\lambda)$ has area $O(|G(s,t)| - 1) = O(d \cdot |G(s,t)|^{1+p})$.

If $G(s,t)$ contains at least two internal faces, then we give the proof for the case where $(u_1,t)$ is the dual of an edge of $P$ (the proof for the case where $(s,u_1)$ is the dual of an edge of $P$ is similar).

We give the proof for the general case where $2 \leqslant b \leqslant k-1$, $m \geqslant 2$, $1 \leqslant c \leqslant m-1$, and Case II applies to $G(u_b,u_{b+1})$. The proofs for the special cases where where $b=1$, or $b=k$, or $m<2$, or $c=0$, or $c=m$, or Case I applies to $G(u_b,u_{b+1})$ are similar.

We will use induction to prove that $\Gamma(s,t,\lambda)$ has area $O(d \cdot |G(s,t)|^{1+p})$. We give the proof for $\Gamma(s,t,1)$. The proofs for $\Gamma(s,t,0)$ and $\Gamma(s,t,-1)$ are similar.

See Fig. 20. We have that $y(w_{h+1}) - y(s) = (y(w_{h+1}) - y(v_m)) + (y(v_m) - y(u_{b+1})) + (y(u_{b+1} - y(s)) = (m-c+1) + (\theta+1) + 2 = m-c+4+\theta = m-c+4+\max\{\max_{b+1\leqslant i\leqslant k-1} H(\Gamma(u_i,u_{i+1},0)), \max_{h+1\leqslant i\leqslant q-1} H(\Gamma(w_i,w_{i+1},0)), \max_{G(z_i,z_j)\in L} H(\Gamma(z_i,z_j,0))\}$.

$$\begin{aligned}
H(\Gamma(s,t,1)) \leqslant\ & y(w_{h+1}) - y(s) \\
& + \max\{H(\Gamma(s,u_1,1)), H(\Gamma(u_1,u_2,1)), \max_{2\leqslant i\leqslant b-1} H(\Gamma(u_1,u_{i+1},0)), \\
& \max_{1\leqslant i\leqslant c-1} H(\Gamma(v_i,v_{i+1},0)), \max_{1\leqslant i\leqslant h-1} H(\Gamma(w_i,w_{i+1},0)), \\
& \max_{G(z_j,z_i)\in R} H(\Gamma(z_j,z_i,0))\} \leqslant m-c+4 \\
& + \max\{\max_{b+1\leqslant i\leqslant k-1} H(\Gamma(u_i,u_{i+1},0)), \max_{h+1\leqslant i\leqslant q-1} H(\Gamma(w_i,w_{i+1},0)), \\
& \max_{G(z_i,z_j)\in L} H(\Gamma(z_i,z_j,0))\} + \max\{H(\Gamma(s,u_1,1)), \\
& H(\Gamma(u_1,u_2,1)), \max_{2\leqslant i\leqslant b-1} H(\Gamma(u_1,u_{i+1},0)), \max_{1\leqslant i\leqslant c-1} H(\Gamma(v_i,v_{i+1},0)), \\
& \max_{1\leqslant i\leqslant h-1} H(\Gamma(w_i,w_{i+1}),0), \max_{G(z_j,z_i)\in R} H(\Gamma(z_j,z_i,0))\}.
\end{aligned}$$

We can express $H$ as a function of the size of $T(s,t)$, i.e., of $|T(s,t)|$. Hence, the above equation can be rewritten as

$$\begin{aligned}
H(|T(s,t)|) \leqslant\ & m-c+4 \\
& + \max\{\max_{b+1\leqslant i\leqslant k-1} H(|T(u_i,u_{i+1})|), \max_{h+1\leqslant i\leqslant q-1} H(|T(w_i,w_{i+1})|), \\
& \max_{G(z_i,z_j)\in L} H(|T(z_i,z_j)|)\} + \max\{H(|T(s,u_1)|), \\
& H(|T(u_1,u_2|), \max_{2\leqslant i\leqslant b-1} H(|T(u_i,u_{i+1}|), \max_{1\leqslant i\leqslant c-1} H(|T(v_i,v_{i+1})|), \\
& \max_{1\leqslant i\leqslant h-1} H(|T(w_i,w_{i+1})|), \max_{G(z_j,z_i)\in R} H(|T(z_j,z_i)|)\}.
\end{aligned}$$

Each $v_i$, where $c \leqslant i \leqslant m$, is a neighbor of $u_b$ or $u_{b+1}$ (or both), hence $m-c \leqslant 2d$.

Let $\alpha$ (respectively, $\beta$) be the left (respectively, right) subtree of $P$ with the maximum size.

$T(s,u_1)$ is a right subtree of $P$. Hence, $|T(s,u_1)| \leqslant |\beta|$. Likewise, for each $i$, where $1 \leqslant i \leqslant b-1$, $|T(u_i,u_{i+1})| \leqslant |\beta|$, for each $i$, where $1 \leqslant i \leqslant h-1$, $|T(w_i,w_{i+1})| \leqslant |\beta|$, and for each $G(z_j,z_i) \in R$, $|T(z_j,z_i)| \leqslant |\beta|$.

Using a similar reasoning gives that for each $i$, where $b+1 \leqslant i \leqslant k-1$, $|T(u_i,u_{i+1})| \leqslant |\alpha|$, for each $i$, where $h+1 \leqslant i \leqslant q-1$, $|T(w_i,w_{i+1})| \leqslant |\alpha|$, and for each $G(z_i,z_j) \in L$, $|T(z_i,z_j)| \leqslant |\alpha|$.

Hence, from the above equation we get that

$$H(|T(s,t)|) \leqslant 2d + 4 + H(|\alpha|) + H(|\beta|).$$

From Lemma 1, $|\alpha|^p + |\beta|^p \leqslant (1-\delta)|T(s,t)|^p$, for some constant $\delta > 0$, and $p = 0.48$. From Lemma 2, it follows that $H(|T(s,t)|) = O((2d+4) \cdot |T(s,t)|^p) = O(d \cdot |T(s,t)|^p)$. Since $|T(s,t)| = O(|G(s,t)|)$, $H(\Gamma(s,t,1)) = H(|T(s,t)|) = O(d \cdot |G(s,t)|^p)$.

From Lemma 5, $\Gamma(s,t,1)$ is a trapezoidal drawing. Hence, $W(\Gamma(s,t,1)) = |G(s,t)| - 1$. Hence, the area of $\Gamma(s,t,1)$ is equal to $W(\Gamma(s,t,1)) \cdot H(\Gamma(s,t,1)) = (|G(s,t)| - 1) \cdot \mathrm{O}(d \cdot |G(s,t)|^p) = \mathrm{O}(d \cdot |G(s,t)|^{1+p})$. $\square$

Lemma 8 summarizes Lemmas 5–7.

**Lemma 8.** *Drawing $\Gamma(s,t,\lambda)$ is trapezoidal, and has $\mathrm{O}(d \cdot |G(s,t)|^{1+p})$ area, where $p = 0.48$, and $d$ is the degree of $G(s,t)$. Moreover, if we have already constructed $T(s,t)$, and computed the size of the subtree rooted at each vertex $\tau$ of $T(s,t)$, and stored it in $\tau$, then $\Gamma(s,t,\lambda)$ can be constructed in $\mathrm{O}(|G(s,t)|)$ time.*

## 7. Drawing an outerplanar graph

We present our main theorem now.

**Theorem 1.** *Let $G$ be an outerplanar graph with degree $d$ and $n$ vertices, then $G$ admits a planar straight-line grid drawing with area $\mathrm{O}(dn^{1.48})$ in $\mathrm{O}(n)$ time.*

**Proof.** If $G$ contains a cutvertex, then determine all the biconnected components of $G$. This can be done easily in $\mathrm{O}(n)$ time using depth-first search [3]. Next, for each biconnected component, construct a (planar) embedding where all the vertices are on the external face. This can be done in $\mathrm{O}(|B'|)$ time for each biconnected component $B'$, as shown in [14]. Hence, constructing the embeddings for all the biconnected components takes $\mathrm{O}(n)$ time in total.

Simply *patch* these embeddings at the corresponding cutvertices to obtain a (planar) embedding of $G$ where all the vertices are on the external face. This can be done easily in $\mathrm{O}(n)$ time. Convert this embedding into a maximal outerplane graph $G^*$ by inserting sufficient number of edges as follows:

- First convert it into a biconnected outerplane graph $G^+$ in $\mathrm{O}(n)$ time by using an algorithm of Kant [11] such that at most $\mathrm{O}(n)$ edges are inserted and the degree of a vertex increases by at most two (this algorithm is called *Algorithm Biconnect* in [11]). Thus, $|G^+| = \mathrm{O}(n)$, and the degree of $G^+$ is at most $d + 2 = \mathrm{O}(d)$.
- For each internal face $f$ of $G^+$ do the following: Let $k$ be the number of vertices on the boundary of $f$. Let $\vartheta_1, \vartheta_2 \ldots, \vartheta_k$ be the counterclockwise sequence of the vertices on the boundary of $f$. Note that $\vartheta_k$ is a neighbor of $\vartheta_1$. If $k$ is an even number, then for each $\vartheta_i$, where $2 \leqslant i \leqslant (k/2) - 1$, insert two edges $(\vartheta_i, \vartheta_{k-i+2})$ and $(\vartheta_i, \vartheta_{k-i+1})$ in the interior of $F$, and for vertex $\vartheta_{k/2}$ insert only one edge $(\vartheta_{k/2}, \vartheta_{k/2+2})$. If $k$ is an odd number, then for each $\vartheta_i$, where $2 \leqslant i \leqslant \lfloor k/2 \rfloor$, insert two edges $(\vartheta_i, \vartheta_{k-i+2})$ and $(\vartheta_i, \vartheta_{k-i+1})$ in the interior of $F$. For example, if $k = 8$, then the inserted edges are $(\vartheta_2, \vartheta_8)$, $(\vartheta_2, \vartheta_7)$, $(\vartheta_3, \vartheta_7)$, $(\vartheta_3, \vartheta_6)$, $(\vartheta_4, \vartheta_6)$, and if $k = 9$, then the inserted edges are $(\vartheta_2, \vartheta_9)$, $(\vartheta_2, \vartheta_8)$, $(\vartheta_3, \vartheta_8)$, $(\vartheta_3, \vartheta_7)$, $(\vartheta_4, \vartheta_7)$, $(\vartheta_4, \vartheta_6)$.The insertion of these edges will *triangulate* $f$, i.e., partition it into $k - 2$ faces, each bounded by a 3-cycle, i.e., a cycle consisting of exactly three edges.

The resulting graph $G^*$ is a maximal outerplane graph because it is biconnected and each internal face is bounded by a 3-cycle. The conversion from $G^+$ to $G^*$ clearly takes only $\mathrm{O}(|G^+|) = \mathrm{O}(n)$ time. The number of faces of $G^+$ whose boundary can contain a vertex $\vartheta$ is less than or equal to the degree of $\vartheta$. We increase the degree of $\vartheta$ by at most 2 when we triangulate a face whose boundary contains it. Hence, converting $G^+$ into $G^*$ by triangulating its faces can at most triple the degree of $\vartheta$. Thus, the degree of $G^*$ is still $\mathrm{O}(d)$.

Summarizing, the total number of edges inserted into $G$ to obtain $G^*$ is $\mathrm{O}(n)$, and the overall time taken is $\mathrm{O}(n)$. Moreover, $|G^*| = \mathrm{O}(n)$, and the degree $d^*$ of $G^*$ is $\mathrm{O}(d)$.

Arbitrarily select two adjacent vertices $s$ and $t$ of $G^*$, such that the edge $(s,t)$ is on the external face and $s$ precedes $t$ in the counterclockwise ordering of the vertices on the external face of $G$. Construct the dual tree $T^*(s,t)$ of $G^*(s,t)$ using Lemma 4 in $\mathrm{O}(|G^*(s,t)|) = \mathrm{O}(n)$ time. Using a standard post-order traversal of $T^*(s,t)$ in $\mathrm{O}(|G^*(s,t)|) = \mathrm{O}(n)$ time, compute the size of the subtree rooted at each vertex $\tau$ of $T^*(s,t)$, and store it in $\tau$.

Let $p = 0.48$ be a number. Using Lemma 8 construct a (trapezoidal) planar straight-line grid drawing $\Gamma^*(s,t,0)$ of $G^*(s,t)$ with $\mathrm{O}(d^* \cdot |G^*(s,t)|^{1+p}) = \mathrm{O}(d \cdot n^{1.48})$ area in $\mathrm{O}(|G^*(s,t)|) = \mathrm{O}(n)$ time.

Finally, obtain a planar straight-line grid drawing of $G$ by removing from $\Gamma^*(s,t,0)$ the edges that were inserted into $G$ to obtain $G^*$. $\square$

The following corollary is immediate.

**Corollary 1.** *Let G be an outerplanar graph with n vertices and degree d*, *where* $d = \mathrm{o}(n^{0.52})$, *then G admits a planar straight-line grid drawing with* $\mathrm{o}(n^2)$ *area in* $\mathrm{O}(n)$ *time*.

## 8. Conclusion and open problems

We have shown that a large category of outerplanar graphs, namely, those with degree $d$ such that $d = \mathrm{o}(n^{0.52})$, where $n$ is the number of vertices in the graph, admit planar straight-line grid drawings with sub-quadratic areas in $\mathrm{O}(n)$ time (see Corollary 1).

It would be interesting to see, if the above result can be extended to a larger category of outerplanar graphs. The above result uses a particular tree-drawing strategy of Chan [2] that draws the spine of a tree completely straight. Using this strategy leads to a *beam*-like drawing for the faces corresponding to $M$, where the vertices are placed on two rows, and the faces are drawn as triangles connecting these two rows (see Fig. 19). Chan [2] has presented another more sophisticated algorithm for drawing an $n$-vertex tree with area $\mathrm{O}(n^{1+\varepsilon})$, where $\varepsilon > 0$ is an arbitrary constant. Unfortunately, this algorithm does not draw the spine of the tree completely straight. Using this algorithm will not give the beam-like drawing for the faces corresponding to $M$, which may make it more difficult to ensure the planarity of $\Gamma(s, t, \lambda)$. Still, it would be interesting to see, if the algorithm could be somehow used to obtain a more area-efficient drawing of an outerplanar graph. We leave it as an open problem at this juncture.

We mention a few other interesting open problems:

- In the drawing constructed by our algorithm, all the vertices are placed on the external face. Is it possible to construct a drawing with smaller area, if we are allowed to change the embedding, such that not all the vertices are placed on the external face?
- The aspect ratio, i.e., the ratio of width and height, of the drawings constructed by our algorithm is $\mathrm{O}(n/(dn^{0.48}))$. Is it possible to construct a drawing with the same (or better) area as the drawing constructed by our algorithm, but with a better aspect ratio, closer to $\mathrm{O}(1)$?
- Can we prove any non-trivial lower bound on the area-requirement of a planar straight-line grid drawing of an outerplanar graph?
- Are there other interesting categories of planar graphs that admit planar straight-line grid drawings with sub-quadratic area?

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